ENGINEERING MATH - II

2ND SEMESTER

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INTEGRATION

Def^n:- After studying differentiation it is natural to study its Inverse process. This process is called Integration.

- **Antiderivative:-**

 If $g(x)$ is the derivative of $f(x)$, then $f(x)$ is said to be antiderivative or integral of $g(x)$.

 Ex:- $\frac{d}{dx}(\log x) = \frac{1}{x}$

 i.e derivative of $\log x = \frac{1}{x}$

 $\therefore$ Antiderivative of $\frac{1}{x} = \log x$ &

$$\int \frac{1}{x} \cdot dx = \log x.$$

- Integral Calculus:- The branch of calculus which studies about integration & its application is called Integral calculus.

→ Integral can be represented by Summation.
 & also an elongated 'S' (for summation) is used to denote Integration.

- Again a constant always exists for an antiderivative

 Ex:- $\frac{d}{dx}(\log x) = \frac{1}{x} + 0$

 $\frac{d}{dx}(\log x + 1) = \frac{1}{x} \Rightarrow \int \frac{1}{x} \cdot dx = \log x + 1$

 $\frac{d}{dx}(\log x + 5) = \frac{1}{x} \Rightarrow \int \frac{1}{x} \cdot dx = \log x + c$

 $\therefore \int \frac{1}{x} \cdot dx = \log x + c$

So $\boxed{\int f(x)\cdot dx = F(x)+c}$

where $\int$ → Symbol of Integration

$f(x)$ - any function x.

dx - integration with respect to x

$F(x)$ - Integral value

c - constant of Integration.

TYPES OF INTEGRALS : —

- Integrals are of Two types
  1) Indefinite Integral
  2) Definite Integral

1) Indefinite Integral :—

Let $f(x)$ be a function. Then the family of all its primitives (or antiderivatives) is called indefinite Integral of $f(x)$ & it is denoted by $\int f(x).dx$.

i.e, $\frac{d}{dx}(\phi(x)+c) = f(x) \Leftrightarrow \int f(x).dx = \phi(x)+c$

2) Definite Integral :—

Let $g(x)$ be the primitive or antiderivative of a continuous function $f(x)$ defined on $[a,b]$ i.e $\frac{d}{dx}\{g(x)\} = f(x)$

Then the definite integral of $f(x)$ over $[a,b]$ is denoted by $\int_a^b f(x).dx = g(b) - g(a)$

where a, b → limits of Integration

b called Upper limit

a called lower limit

$[a,b]$ → Interval of Integration

• **Application:-**

- Integration is used to find the areas under lines & curves. Entire area is divided into infinitesimally small regions, area of each region is found & added to get the entire area.

• **FUNDAMENTAL INTEGRAL FORMULAS:-**

1. $\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c \;,\; n \neq -1$

2. $\int \frac{1}{x} \cdot dx = \log_e |x| + c$

3. $\int e^x \cdot dx = e^x + c$

4. $\int a^x \cdot dx = \int \frac{a^x}{\log_e a} + c$

5. $\int \sin x \cdot dx = -\cos x + c$

6. $\int \cos x \cdot dx = \sin x + c$

7. $\int \sec^2 x \cdot dx = \tan x + c$

8. $\int \operatorname{cosec}^2 x \cdot dx = -\cot x + c$

9. $\int \sec x \cdot \tan x \cdot dx = \sec x + c$

10. $\int \operatorname{cosec} x \cdot \cot x \cdot dx = -\operatorname{cosec} x + c$

11. $\int \cot x \cdot dx = \ln |\sin x| + c$

12. $\int \tan x \cdot dx = -\ln |\cos x| + c$

13. $\int \sec x \cdot dx = \log |\sec x + \tan x| + c$

14. $\int \operatorname{cosec} x \cdot dx = \log |\operatorname{cosec} x - \cot x| + c$

15. $\int \frac{1}{\sqrt{a^2-x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

16. $\int \frac{-1}{\sqrt{a^2-x^2}} \cdot dx = \cos^{-1}\left(\frac{x}{a}\right) + c$

17. $\int \frac{1}{a^2+x^2} \cdot dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

18. $\int \frac{-1}{a^2+x^2} \cdot dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + c$

19. $\int \frac{1}{x\sqrt{x^2-a^2}} \cdot dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$

20. $\int -\frac{1}{x\sqrt{x^2-a^2}} \cdot dx = \frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + c$

21. $\int K \cdot dx = Kx + c$, K – constant

22. $\int \sqrt{x} \cdot dx = \frac{2}{3}x^{\frac{3}{2}} + c$

23. $\int \frac{1}{\sqrt{x}} \cdot dx = 2\sqrt{x} + c$

* Algebra of Integration: –

(i) $\int [f(x) + g(x)] \cdot dx = \int f(x) dx + \int g(x) \cdot dx$

(ii) $\int \lambda f(x) \cdot dx = \lambda \int f(x) \cdot dx$, for some constant λ.

(iii) $\int [\lambda_1 f_1(x) + \lambda_2 f_2(x) + \dots + \lambda_n f_n(x)] \, dx$

$= \lambda_1 \int f_1(x) \cdot dx + \lambda_2 \int f_2(x) \cdot dx + \dots + \lambda_n \int f_n(x) \cdot dx$

(iv) $\frac{d}{dx}\left(\int f(x).dx\right) = f(x)$.

i.e the differentiation of an integral is the integral itself or differentiation & integration are inverse operations.

• problems :—

Ex:-1 $\int 4x^5 \cdot dx = 4\int x^5 \cdot dx = 4\left(\frac{x^{5+1}}{5+1}\right) + c = \frac{4}{6}x^6 + c = \frac{2}{3}x^6 + c$

Eg.2: $\int 2\sin x \cdot dx = 2\int \sin x \cdot dx = -2\cos x + c$

Eg.3. $\int 3^{x+2} \cdot dx = \int 3^x \cdot 3^2 \cdot dx = 3^2\int 3^x \cdot dx$

$$= 9\int 3^x \cdot dx$$

$$= 9\left(\frac{3^x}{\log 3}\right) + c$$

Eg.4: $\int \frac{1}{2}\sec^2 x \cdot dx = \frac{1}{2}\int \sec^2 x \cdot dx = \frac{1}{2}\tan x + c$

Eg.4: $\int (x^6 + x^2 + x + 1)\, dx$

$$= \int x^6 \cdot dx + \int x^2 \cdot dx + \int x \cdot dx + \int 1 \cdot dx$$

$$= \frac{x^{6+1}}{6+1} + \frac{x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1} + x + C \qquad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$= \frac{x^7}{7} + \frac{x^3}{3} + \frac{x^2}{2} + x + c$$

Eg-5 $\int e^{3x} \cdot dx = \int (e^3)^x \cdot dx = \frac{(e^3)^x}{\log e^3} + K = \frac{e^{3x}}{3} + K$

eg.6: $\int \left(\frac{x^4}{x^2+1}\right) \cdot dx = \int \left(x^2 - 1 + \frac{1}{x^2+1}\right) dx$

$$= \int x^2 dx - \int 1 \cdot dx + \int \frac{1}{x^2+1} \cdot dx$$

$$= \frac{x^3}{3} - x + \tan^{-1}x + K$$

eg.7:- $\int 6x^3(x+5)^2 \cdot dx$

$$= \int 6x^3\left(x^2+10x+25\right)\cdot dx$$

$$= \int \left(6x^5 + 60x^4 + 150x^3\right) dx$$

$$= \int 6x^5 \cdot dx + \int 60x^4 \cdot dx + \int 150x^3 \cdot dx$$

$$= 6\int x^5 \cdot dx + 60\int x^4 \cdot dx + 150\int x^3 \cdot dx$$

$$= 6 \times \frac{x^{5+1}}{5+1} + 60 \cdot \frac{x^{4+1}}{4+1} + 150\ \frac{x^{3+1}}{3+1} + c$$

$$= \frac{\cancel{6}x^6}{\cancel{6}} + 60\frac{x^5}{5} + 150 \times \frac{x^4}{4} + c$$

$$= x^6 + 12x^5 + \frac{75}{2}x^4 + c$$

eg.8:- $\int \frac{dx}{\cos^2 x} = \int -\sec^2 x \cdot dx = -\int \sec^2 x \cdot dx$

$$= -\tan x + c$$

eg.9:- $\int (e^x+2)dx = \int e^x \cdot dx + \int 2 \cdot dx$

$$= e^x + 2x + c$$

eg.10 $\int \frac{x^4+x^3+x^2+x+2}{x^2+1} \cdot dx$

$$= \int \frac{(x^4+x^2)+(x^3+x)+2}{x^2+1} \cdot dx$$

$$= \int \frac{x^4+x^2}{x^2+1}dx + \int \frac{x^3+x}{x^2+1}.dx + \int \frac{2}{x^2+1}dx$$

$$= \int \frac{x^4(\cancel{x^2+1})}{\cancel{x^2+1}}dx + \int \frac{x(\cancel{x^2+1})}{\cancel{x^2+1}}dx + 2\int \frac{1}{x^2+1}.dx$$

$$= \int x^4.dx + \int x.dx + 2\int \frac{1}{x^2+1}.dx$$

$$= \frac{x^{4+1}}{4+1} + \frac{x^{1+1}}{1+1} + 2\tan^{-1}x + c$$

$$= \frac{x^5}{5} + \frac{x^2}{2} + 2\tan^{-1}x + c$$

eg 11:- $\int \frac{1-\sin^3 x}{\sin^2 x}.dx$

$$= \int \frac{1}{\sin^2 x}.dx - \int \frac{\sin^{\cancel{3}} x}{\cancel{\sin^2 x}}.dx$$

$$= \int \operatorname{cosec}^2 x.dx - \int \sin x.dx$$

$$= -\cot x + \cos x + C$$

eg 12: $\int \sqrt{1-\cos 2x}.dx$

$$= \int \sqrt{2\sin^2 x}.dx \quad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right.$$

$$= \sqrt{2}\int \sqrt{\sin^2 x}.dx$$

$$= \sqrt{2}\int \sin x.dx$$

$$= \sqrt{2}\cos x + C$$

eg 13:> $\displaystyle\int \frac{\cos^4 x - \sin^4 x}{\cos x - \sin x}\,.dx$

$$= \int \frac{(\cos^2 x)^2 - (\sin^2 x)^2}{\cos x - \sin x}\,.dx$$

$$= \int \frac{(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)}{\cos x - \sin x}\,.dx \quad \left[\because a^2 - b^2 = (a+b)(a-b)\right]$$

$$= \int \frac{1 \times (\cos^2 x - \sin^2 x)}{\cos x - \sin x}\,.dx \quad \left[\because \cos^2 x + \sin^2 x = 1\right]$$

$$= \int \frac{(\cos x + \sin x)\cancel{(\cos x - \sin x)}}{\cancel{\cos x - \sin x}}\,dx \quad \left[\because a^2 - b^2 = (a+b)(a-b)\right]$$

$$= \int (\cos x + \sin x)\,dx$$

$$= \int \cos x\,.dx + \int \sin x\,.dx$$

$$= \int \cos x\,.dx - \int -\sin x\,.dx$$

$$= \sin x - \cos x + C$$

eg-14:- $\displaystyle\int \frac{1 - \cos 2x}{1 + \cos 2x}\,dx$

$$= \int \frac{\cancel{2}\sin^2 x}{\cancel{2}\cos^2 x}\,.dx \quad \left[\begin{array}{l} \because \sin^2 x = \dfrac{1 - \cos 2x}{2} \\ \cos^2 x = \dfrac{1 + \cos 2x}{2} \end{array}\right]$$

$$= \int \tan^2 x\,.dx$$

$$= \int (\sec^2 x - 1)\,dx$$

$$= \int \sec^2 x\,.dx - \int 1\,.dx$$

$$= \tan x - x + C$$

eg.15 :- $\int \sqrt{1+\sin 2x}\cdot dx$

$= \int \sqrt{(\sin^2 x + \cos^2 x) + 2\sin x\cdot\cos x}\cdot dx$

$= \int \sqrt{(\sin x + \cos x)^2}\cdot dx$

$= \int \sin x + \cos x \cdot dx$

$= \int \sin x\, dx + \int \cos x \cdot dx$

$= -\cos x + \sin x + C$

$= \sin x - \cos x + C$

eg-16 :- $\int \frac{\sin x}{\cos^2 x}\cdot dx$

$= \int \frac{\sin x}{\cos x\cdot\cos x}\cdot dx$

$= \int \tan x \cdot \sec x \cdot dx$

$= \sec x + C$

eg-17 :- $\int (x^2+\sqrt{x})^2\cdot dx$

$= \int \left(x^4 + (\sqrt{x})^2 + 2\cdot x^2\cdot\sqrt{x}\right)\cdot dx$

$= \int x^4 + x + 2x^{5/2}\cdot dx$

$= \int \left(x^4 + 2x^{5/2} + x\right) dx$

$= \int x^4\cdot dx + \int 2x^{5/2}\cdot dx + \int x\cdot dx$

$= \frac{x^{4+1}}{4+1} + 2\frac{x^{5/2+1}}{5/2+1} + \frac{x^2}{2} + C$

$= \cancel{\frac{x^5}{5}} + \cancel{\frac{2x^{3}}{3}} + \cancel{\frac{x^2}{2}} + \cancel{C} = \frac{x^5}{5} +$

METHODS OF INTEGRATION :—

We have the following Methods of Integration

(i) Integration by Substitution

(ii) Integration by parts

(iii) Integration of rational algebraic functions by using partial fractions.

(i) INTEGRATION BY SUBSTITUTION :—

When the integral is not in the standard form it can be transformed to integrable form by a suitable substitution. The Integral

$\int f(g(x))\, g'(x) \cdot dx$ can be converted to

$= \int f(t) \cdot dt$ where $g(x) = t$

$= F(t) + K$

There is no direct formula for substitution. Keen observation of the form of the Integrand will help choosing appropriate substitution.

Ex • $\int f(ax+b)\, dx$

Let $ax + b = t$

$\Rightarrow d(ax+b) = dt$

$\Rightarrow a\, dx = dt$

$\Rightarrow dx = \frac{1}{a} dt$

substituting $ax+b=t$ & $dx=\frac{1}{a}dt$, we get

$$I=\int f(ax+b)\,dx=\int f(t)\,\frac{1}{a}\cdot dt=\frac{1}{a}\int f(t)\,dt$$

$$=\frac{1}{a}\frac{t^2}{2}=\frac{1}{a}\,ax+$$

$$=\frac{1}{a}\phi(t)=\frac{1}{a}\phi(ax+b)$$

Formula's Using Substitution: –

1. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}+c$, $n\neq -1$

2. $\int \frac{1}{ax+b}\cdot dx=\frac{1}{a}\log|ax+b|+c$

3. $\int e^{ax+b}\cdot dx=\frac{1}{a}e^{ax+b}+C$

4. $\int a^{bx+c}\cdot dx=\frac{1}{b}\frac{a^{bx+c}}{\log a}+C$, $a>0$ & $a\neq 1$

5. $\int \sin(ax+b)\,dx=\frac{-1}{a}\cos(ax+b)+c$

6. $\int \cos(ax+b)\,dx=\frac{1}{a}\sin(ax+b)+c$

7. $\int \sec^2(ax+b)\,dx=\frac{1}{a}\tan(ax+b)+c$

8. $\int \operatorname{cosec}^2(ax+b)\,dx=\frac{-1}{a}\cot(ax+b)+C$

9. $\int \sec(ax+b)\tan(ax+b)\,dx=\frac{1}{a}\sec(ax+b)+C$

10. $\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx=\frac{-1}{a}\operatorname{cosec}(ax+b)+C$

11. $\int \tan(ax+b)\cdot dx=\frac{1}{a}\log|\cos(ax+b)|+C$

12. $\int \cot(ax+b)\,dx=\frac{1}{a}\log|\sin(ax+b)|+c$

(13) $\int \sec(ax+b)\,dx = \frac{1}{a}\log|\sec(ax+b)+\tan(ax+b)| + c$

(14) $\int \operatorname{cosec}(ax+b)\,dx = \frac{1}{a}\log|\operatorname{cosec}(ax+b) - \cot(ax+b)| + c$

• <u>Problems</u> : —

① $\int e^{2x-3}\,.dx = \frac{1}{2} \times e^{2x-3} + C$

② $\int e^{3x+2}\,.dx = \frac{1}{3\log a} \times a^{3x+2} + C$

③ $\int \frac{\sin 4x}{\sin 2x}\,.dx = \int \frac{2\sin 2x \cdot \cos 2x}{\sin 2x}\,.dx$

$= 2\int \cos 2x\,.dx$

$= \frac{2}{2}\sin 2x + c$

$= \sin 2x + C$

④ $\int \sqrt{1+\sin x}\,.dx \;,\; 0 < x < \frac{\pi}{2}$

$I = \int \sqrt{1+\sin x}\,.dx$

$= \int \sqrt{\sin^2\frac{x}{2} + \cos^2 x/_2 + 2\sin x/_2 \cdot \cos x/_2}\,.dx$

$= \int \sqrt{\left(\sin \frac{x}{2} + \cos x/_2\right)^2}\,.dx$

$= \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) dx$

$$= \int \cos\frac{x}{2}\cdot dx + \int \sin\frac{x}{2}\cdot dx$$

$$= 2\sin\frac{x}{2} + 2\cos\frac{x}{2} + c$$

$$= 2\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + c$$

Ex-5: $\int \frac{1}{\sqrt{3x+4} - \sqrt{3x+1}}$

Let $I = \int \frac{1}{\sqrt{3x+4} - \sqrt{3x+1}} \cdot dx$

$$= \int \frac{\sqrt{3x+4} + \sqrt{3x+1}}{(\sqrt{3x+4} - \sqrt{3x+1})(\sqrt{3x+4} + \sqrt{3x+1})} \cdot dx$$

$$= \int \frac{\sqrt{3x+4} + \sqrt{3x+1}}{(3x+4) - (3x+1)} \cdot dx$$

$$= \frac{1}{3} \int \left\{\sqrt{3x+4} + \sqrt{3x+1}\right\} \cdot dx$$

$$= \frac{1}{3} \int \sqrt{3x+4} \cdot dx + \frac{1}{3} \int \sqrt{3x+1} \cdot dx$$

$$= \frac{1}{3} \left\{ \frac{(3x+4)^{3/2}}{3 \times \frac{3}{2}} \right\} + \frac{1}{3} \left\{ \frac{(3x+1)^{3/2}}{3 \times \frac{3}{2}} \right\} + c$$

$$= \frac{2}{27} \left\{ (3x+4)^{3/2} + (3x+1)^{3/2} \right\} + C$$

Ex-6:- $\int \frac{8^{1+x} + 4^{1-x}}{2^x} \cdot dx$

$$I = \int \frac{8^{1+x} + 4^{1-x}}{2^x} \cdot dx = \int \frac{2^{3x+3} + 2^{2-2x}}{2^x} \cdot dx$$

$$= \int \left(2^{2x+3} + 2^{2-3x}\right) dx$$

$$= \frac{2^{2x+3}}{2\log 2} + \frac{2^{2-3x}}{(-3)\log 2} + C$$

Ex-7: $\int \sec^2(7-4x)\,dx = \frac{-1}{4}\tan(7-4x) + c$

#

Integration of Trigonometric function of the form

$$\int \sin mx \cdot \cos nx \cdot dx \ , \ \int \sin mx \cdot \sin nx \cdot dx \ ,$$

$$\& \ \int \cos mx \cdot \cos nx \cdot dx$$

- To evaluate this type of integrals we use the following trigonometrical identities to express the products into sums

$$2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$$

- Problems:-

Ex-1: $\int \sin 4x \ \cos 3x \cdot dx$

$$I = \int \sin 4x \cdot \cos 3x \cdot dx$$
$$= \frac{1}{2}\int 2\sin 4x \ \cos 3x \cdot dx$$
$$= \frac{1}{2}\int (\sin 7x + \sin x)\,dx$$
$$= \frac{1}{2}\left\{ -\frac{\cos 7x}{7} - \cos x \right\} + c$$

Ex-2: $\int \sin 3x \cdot \sin 2x \cdot dx$

$$I = \int \sin 3x \cdot \sin 2x \ dx$$
$$= \frac{1}{2}\int 2\sin 3x \cdot \sin 2x \cdot dx$$

$$= \frac{1}{2}\int \left\{ (\cos x - \cos 5x) \right\} dx$$

$$= \frac{1}{2}\left\{ \sin x - \frac{\sin 5x}{5} \right\} + C$$

Ex-3: $$I = \int \sin x \cdot \sin 2x \cdot \sin 3x \cdot dx$$

$$= \frac{1}{2}\int \left(2 \sin 2x \cdot \sin x\right) \sin 3x \cdot dx$$

$$\cancel{= \frac{1}{2}\int \left(\cos 6x + \cos 2x\right) \sin 3x \cdot dx}$$

$$= \frac{1}{2}\int \left(\cos x - \cos 3x\right) \sin 3x \cdot dx$$

$$= \frac{1}{2}\int \left(2 \sin 3x \cos x - 2 \sin 3x \cos 3x\right) dx$$

$$= \frac{1}{4}\int \left(\sin 4x + \sin 2x - \sin 6x\right) dx$$

$$= \frac{1}{4}\left\{ -\frac{\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6} \right\} + C$$

Ex-4: $$I = \int \frac{\sin 4x}{\sin x} \cdot dx$$

$$= \int \frac{2 \sin 2x \cdot \cos 2x}{\sin x} dx$$

$$= \int \frac{4 \cancel{\sin x} \cdot \cos x \cdot \cos 2x}{\cancel{\sin x}} \cdot dx$$

$$= \int 4 \cos x \cdot \cos 2x \cdot dx$$

$$= 2\int 2 \cos x \cdot \cos 2x \, dx$$

$$= 2\int \left(\cos 3x + \cos x\right) dx$$

$$= 2\left\{ \frac{\sin 3x}{3} + \sin x \right\} + C$$

$$= 2\left\{ \cancel{\frac{\sin 3x}{3}} + \cancel{\sin x} \right\} + C$$

Ex-4 (ii) $\int \sin^2 x \cdot dx$

$$= \int \frac{1}{2}(1 - \cos 2x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

$$= \frac{1}{4}(2x - \sin 2x) + C$$

Ex 5: $\int \sin^3 x \cdot \cos^3 x \cdot dx$

$$= \frac{1}{8}\int (2\sin x \cos x)^3 \cdot dx$$

$$= \frac{1}{8}\int (\sin 2x)^3\, dx$$

$$= \frac{1}{32}\int 4\sin^3 2x\, dx = \frac{1}{32}\int (3\sin 2x - \sin 6x)\, dx$$

$$= \frac{1}{32}\left[\cancel{\frac{4}{3}}\ \frac{-3}{2}\cos 2x + \frac{1}{6}\cos 6x\right] + C$$

$$= \frac{1}{192}(\cos 6x - 9\cos 2x) + C$$

Evaluation of Integrals of the form $\int \sin^m x \cdot dx$, $\int \cos^m x \, dx$, $m \le 4$

- $\sin^2 x = \dfrac{1-\cos 2x}{2}$
- $\cos^2 x = \dfrac{1+\cos 2x}{2}$
- $\sin 3x = 3\sin x - 4\sin^3 x$
- $\cos 3x = 4\cos^3 x - 3\cos x$

Problems:-

Ex-1: $\int \sin^2 x \cdot dx$

$$= \int \frac{1-\cos 2x}{2} \cdot dx$$

$$= \frac{1}{2}\int 1-\cos 2x \cdot dx$$

$$= \frac{1}{2}\left\{ x - \frac{\sin 2x}{2} \right\} + C$$

Ex-2: $\int \sin^3 x \cdot dx$

$$= \int \frac{3\sin x - \sin 3x}{4} \cdot dx$$

$$= \frac{1}{4}\int (3\sin x - \sin 3x) dx$$

$$= \frac{1}{4}\left\{ -3\cos x + \frac{\cos 3x}{3} \right\} + c$$

Ex-3: $\int \sin^4 x \cdot \cos^4 x \cdot dx$

$$= \frac{1}{16}\int (2\sin x \cos x)^4 dx$$

$$= \frac{1}{16}\int (\sin 2x)^4 \cdot dx$$

$$= \frac{1}{16}\int (\sin^2 2x)^2 \cdot dx$$

$$= \frac{1}{16}\int \left(\frac{1-\cos 4x}{2}\right)^2 \cdot dx$$

$$= \frac{1}{64}\int (1 - 2\cos 4x + \cos^2 4x) dx$$

$$= \frac{1}{64}\int \left\{ 1 - 2\cos 4x + \frac{1+\cos 8x}{2} \right\} dx$$

$$= \frac{1}{128}\int (3 - 4\cos 4x + \cos 8x) dx$$

$$= \frac{1}{128}\left\{ 3x - \sin 4x + \frac{1}{8}\sin 8x \right\} + c$$

Evaluation of Integral of the form $\frac{P(x)}{(ax+b)^n}$, $n \in N$:—

where $P(x)$ is a polynomial.

Problems:-

Ex-1:- $\int \frac{x^3}{(x+2)^4} \, dx$

$$= \int \frac{\{(x+2)-2\}^3}{(x+2)^4} \, dx$$

$$= \int \frac{(x+2)^3 - 8 - 6(x+2)^2 + 12(x+2)}{(x+2)^4} \, dx$$

$$= \int \left\{ \frac{1}{x+2} - \frac{6}{(x+2)^2} + \frac{12}{(x+3)^3} - \frac{8}{(x+2)^4} \right\} \cdot dx$$

$$= \log|x+2| + \frac{6}{x+2} - \frac{6}{(x+2)^2} + \frac{8}{3(x+2)^3} + C$$

Ex-2:- $\int \frac{ax+b}{(cx+d)^2} \cdot dx$

Let $ax+b = \lambda(cx+d) + \mu$

On equating coefficient of like powers of x, we get $a = \lambda c$ & $b = \lambda d + \mu$

$$\Rightarrow \lambda = \frac{a}{c}$$

$$\& \ \mu = \frac{bc-ad}{c}$$

$$= \int \frac{ax+b}{(cx+d)^2} \cdot dx$$

$$= \int \frac{\lambda(cx+d) + \mu}{(cx+d)^2} \cdot dx$$

$$= \lambda \int \frac{1}{cx+d} \cdot dx + \mu \int \frac{1}{(cx+d)^2} \cdot dx$$

$$= \frac{\lambda}{c} \log|cx+d| - \frac{\mu}{c(cx+d)} + C$$

$$= \frac{a}{c^2} \log|cx+d| - \frac{(bc-ad)}{c^2} \times \frac{1}{cx+d} + C$$

<u>Ex-2</u>: $\int \frac{x+2}{(x+1)^2} \cdot dx$

Let $x+2 = \lambda(x+1)+\mu$

On equating the coefficeints of like powers of x on both sides, we get.

$\lambda = 1$ & $2 = \lambda+\mu \Rightarrow \mu = 1$

$$= \int \frac{\lambda(x+1)+\mu}{(x+1)^2} \cdot dx$$

$$= \int \left\{ \frac{\lambda}{x+1} + \frac{\mu}{(x+1)^2} \right\} dx$$

$$= \lambda \int \frac{1}{x+1} \, dx + \mu \int \frac{1}{(x+1)^2} \, dx$$

$$= \lambda \log|x+1| - \frac{\mu}{x+1} + C$$

$$= \log|x+1| - \frac{1}{x+1} + C$$

<u>Ex-3</u>: $\int \frac{x^2}{(a+bx)^2} \cdot dx$

using long division method

$$\frac{x^2}{(a+bx)^2} = \frac{1}{b^2} + \frac{-\frac{2a}{b}x - \frac{a^2}{b^2}}{(bx+a)^2}$$

$$\Rightarrow \frac{x^2}{(a+bx)^2} = \frac{1}{b^2} - \frac{a}{b^2}\left(\frac{2bx+a}{(bx+a)^2}\right)$$

$$\Rightarrow \frac{x^2}{(a+bx)^2} = \frac{1}{b^2} - \frac{a}{b^2}\left\{\frac{2(bx+a)-a}{(bx+a)^2}\right\}$$

$$= \frac{1}{b^2} - \frac{2a}{b^2} \times \frac{1}{bx+a} + \frac{a^2}{b^2} \times \frac{1}{(bx+a)^2}$$

$$\therefore \int \frac{x^2}{(a+bx)^2} \cdot dx$$

$$= \int \left\{ \frac{1}{b^2} - \frac{2a}{b^2} \times \frac{1}{bx+a} + \frac{a^2}{b^2} \times \frac{1}{(bx+a)^2} \right\} \cdot dx$$

$$= \frac{1}{b^2}\int 1 \cdot dx - \frac{2a}{b^2}\int \frac{1}{bx+a} \cdot dx + \frac{a^2}{b^2}\int \frac{1}{(bx+a)^2} \cdot dx$$

$$= \frac{x}{b^2} - \frac{2a}{b^3} \log|bx+a| - \frac{a^2}{b^3} \times \frac{1}{bx+a} + C$$

$$= \frac{1}{b^3}\left\{ bx - 2a\log|bx+a| - \frac{a^2}{bx+a} \right\} + C$$

<u>Ex-3</u>: $\int \frac{x^2+1}{(x+1)^2} \cdot dx$

$$= \int \frac{x^2+1+2x-2x}{(x+1)^2} \cdot dx$$

$$= \int \frac{(x+1)^2 - 2x}{(x+1)^2} \cdot dx$$

$$= \int 1 - \frac{2x}{(x+1)^2} \cdot dx$$

$$= \int 1 \cdot dx - 2\int \frac{x}{(x+1)^2}\, dx$$

$$= \int 1 \cdot dx - 2\int \frac{(x+1)-1}{(x+1)^2} \cdot dx$$

$$= \int 1\cdot dx - 2\int\left\{\frac{1}{x+1} - \frac{1}{(x+1)^2}\right\}dx$$

$$= \int 1\cdot dx - 2\int \frac{1}{x+1}\cdot dx + 2\int \frac{1}{(x+1)^2}\,dx$$

$$= x - 2\log|x+1| - \frac{2}{x+1} + c$$

Evaluation of Integral of the form $\int (ax+b)\sqrt{cx+d}\cdot dx$ & $\int \frac{ax+b}{\sqrt{cx+d}}\cdot dx$: —

Algorithm :—

Step-1 : Let $(ax+b) = \lambda(cx+d)+\mu$

Step-II : find λ & μ equating the coefficients of like powers of x on both sides.

Step-III : Replace $(ax+b)$ by $\lambda(cx+d)+\mu$ ~~we get~~ & integrating we get the value.

Problems :-

Ex-1 : $\int x\sqrt{x+2}\cdot dx$

$$= \int\{(x+2)-2\}\sqrt{x+2}\cdot dx$$

$$= \int\{(x+2)^{3/2} - 2(x+2)^{1/2}\}\,dx$$

$$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c$$

Ex-2 :- $\int (7x-2)\sqrt{3x+2}\cdot dx$

Let $7x-2 = \lambda(3x+2)+\mu$

$3\lambda = 7 \Rightarrow \mu = -\frac{20}{3}$, $\lambda = 7/3$

$$= \int\{\lambda(3x+2)+\mu\}\sqrt{3x+2}\cdot dx$$

$$= \int \left\{ \lambda(3x+2)^{3/2} + \mu(3x+2)^{1/2} \right\} . dx$$

$$= \lambda \left\{ \frac{(3x+2)^{5/2}}{5/2 \times 3} \right\} + \mu \left\{ \frac{(3x+2)^{3/2}}{3 \times \frac{3}{2}} \right\} + c$$

$$= \frac{14}{45}(3x+2)^{5/2} - \frac{40}{27}(3x+2)^{3/2} + C$$

Ex/A:- ∫

Integral of the form $\int \frac{f'(x)}{f(x)} . dx$:—

Theorem :- $\int \frac{f'(x)}{f(x)} . dx = \log\{f(x)\} + C$

Problems:-

(1) $\int \tan x . dx = \int \frac{\sin x}{\cos x} . dx$.

$\cos x = t$ & $dx = -dt/\sin x$

$$I = \int \frac{\sin x}{\cos x} \times \left(-\frac{dt}{\sin x}\right)$$

$$= -\int \frac{1}{t} dt = -\log|t| + c$$

$$= -\log|\cos x| + c$$

$$\boxed{\int \tan x . dx = -\log|\cos x| + c}$$

Results —

i) $\int \tan x \, dx = -\log|\cos x| + c$

ii) $\int \cot x . dx = \log|\sin x| + c$

iii) $\int \sec x . dx = \log|\sec x + \tan x| + c$

&

$\log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

(iv) $\int \operatorname{cosec} x \cdot dx = \log|\operatorname{cosec} x - \cot x| + C$ & $\log\left|\tan\frac{x}{2}\right| + C$

(~~iv~~)

Problem :-

① $\int \frac{1}{\sqrt{1+\cos 2x}} \cdot dx$

$= \int \frac{1}{\sqrt{2\cos^2 x}} \cdot dx$

$= \frac{1}{\sqrt{2}} \int \sec x \cdot dx = \frac{1}{\sqrt{2}} \log|\sec x + \tan x| + c$

② $\int \frac{1}{\sqrt{1-\sin x}} \cdot dx$

$= \int \frac{1}{1-\cos(\pi/2 + 2x)} \cdot dx$

$= \int \frac{1}{2\sin^2(\pi/2 + x)} \cdot dx$

$= \int \frac{1}{\sqrt{2}} \operatorname{cosec}\left(\frac{\pi}{4} + x\right) \cdot dx$

$= \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(\frac{\pi}{4} + x\right) \cdot dx$

$= \frac{1}{\sqrt{2}} \log\left|\tan\left(\frac{\pi}{8} + \frac{x}{2}\right)\right| + C$

③ $\int \frac{\sin(x-a)}{\sin x} \cdot dx$

$= \int \frac{\sin x \cos a - \cos x \cdot \sin a}{\sin x} \cdot dx$

$= \int \cos a \cdot dx - \int \sin a \cot x \cdot dx$

$= \cos a \int 1 \cdot dx - \sin a \int \cot x \cdot dx$

$= x\cos a - \sin a \log|\sin x| + C$

Ex-2: $\int \frac{1}{\sin(x-a)\cdot\cos(x-b)}\cdot dx$

$= \frac{1}{\cos(a-b)}\int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}\cdot dx$

$= \frac{1}{\cos(a-b)}\int \frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cdot\cos(x-b)}\cdot dx$

$= \frac{1}{\cos(a-b)}\int \frac{\cos(x-a)\cos(x-b)+\sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}\cdot dx$

$= \frac{1}{\cos(a-b)}\int\{\cot(x-a)+\tan(x-a)\}dx$

$= \frac{1}{\cos(a-b)}\int \cot(x-a)+\tan(x-b)\}\cdot dx$

$= \frac{1}{\cos(a-b)}\{\log_e|\sin(x-a)|-\log_e|\cos(x-b)|\}+c$

$= \frac{1}{\cos(a-b)}\log_e\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+c$

Ex-3:- $\int \frac{2x+5}{x^2+5x-7}\cdot dx$

Let $x^2+5x-7=t$, then $d(x^2+5x-7)=dt$

$\Rightarrow (2x+5)dx = dt$

$\Rightarrow dx = \frac{dt}{2x+5}$

Putting $x^2+5x-7=t$ & $dx=\frac{dt}{2x+5}$, we get

$I = \int \frac{2x+5}{x^2+5x-7}\cdot dx = \int \frac{1}{t}dt = \log|t|+C$

$= \log|x^2+5x-7|+C$

Ex - 4:- $\int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \cdot dx$

Let $e^{x}+e^{-x}=t$

$\Rightarrow \left(e^{x}-e^{-x}\right) dx = dt$

$\Rightarrow \quad dx = \frac{dt}{e^{x}-e^{-x}}$

$$= \int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \cdot dx = \int \frac{dt}{t} = \log|t| + C = \log|e^{x}+e^{-x}| + c$$

Integration of the form $\int \{f(x)\}^{n} f'(x) \cdot dx$:—

Theorem :— $\int \{f(x)\}^{n} f'(x) \cdot dx = \frac{\{f(x)\}^{n+1}}{n+1}, \ n \neq -1$

Problems:

(1) $\int \sin^{3}x \cos x \cdot dx$

Let $\sin x = t$, then $d(\sin x) = dt$

$\Rightarrow \cos x \cdot dx = dt$

$\Rightarrow dx = \frac{dt}{\cos x}$

Putting $\sin x = t$ & $dx = \frac{dt}{\cos x}$, we get

$$\int \sin^{3}x \cos x \cdot dx = \int t^{3} \cancel{\cos x} \times \frac{dt}{\cancel{\cos x}} = \int t^{3} \cdot dt$$

$$= \frac{t^{4}}{4} + c = \frac{\sin^{4}x}{4} + c$$

(2) $\int \frac{4(\sin^{-1}x)^3}{\sqrt{1-x^2}} \cdot dx$

Let $\sin^{-1}x = t$

$\Rightarrow d(\sin^{-1}x) = dt$

$\Rightarrow \frac{1}{\sqrt{1-x^2}} \cdot dx = dt$

$\Rightarrow dx = \sqrt{1-x^2} \cdot dt$

Putting the above values in the integration we get,

$\int 4t^3 \cdot dt = 4\int t^3 \cdot dt = 4 \times \frac{t^4}{4} + c = t^4 + C = (\sin^{-1}x)^4 + C$

(B1)

Integration of the form $\int \tan^m x \sec^{2n} x \cdot dx$, $\int \cot^m x \operatorname{cosec}^{2n} x \cdot dx$:-

Ex:- $\int \tan^3 x \sec^3 x \cdot dx$

$= \int \tan^2 x \cdot \sec^2 x \cdot (\sec x \cdot \tan x) \cdot dx$

$= \int (\sec^2 x - 1) \sec^2 x \, (\sec x \cdot \tan x) \cdot dx$

Now substituting $\sec x = t$ & $\sec x \cdot \tan x \, dx = dt$, we get

$= \int (t^2 - 1) t^2 \, dt = \int (t^4 - t^2) dt$

$= \frac{t^5}{5} - \frac{t^3}{3} + c$

$= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$

Integral of the form $\int \sin^m x \cos^n x \cdot dx$, $m, n \in N$

Ex:- (1) $\int \sin^3 x \cos^4 x \cdot dx$

Here power of $\sin x$ is odd, So, we substitute

$\cos x = t$

$\Rightarrow -\sin x \cdot dx = dt$

$\Rightarrow dx = -\dfrac{dt}{\sin x}$

$= \int \sin^3 x \; t^4 \left(\dfrac{-dt}{\sin x}\right)$

$= -\int \sin^2 x \cdot t^4 \cdot dt$

$= -\int (1-t^2) t^4 \cdot dt$

$= -\int (t^4 - t^6)\, dt$

$= -\dfrac{t^5}{5} + \dfrac{t^7}{7} + c$

$= -\dfrac{\cos^5 x}{5} + \dfrac{\cos^7 x}{7} + C$

(2) $\int \sin^2 x \cdot \cos^5 x \cdot dt$

Here $\cos x$ is odd, so,

Let $\sin x = t$

$\Rightarrow \cos x \cdot dx = dt$

$\Rightarrow dx = \dfrac{dt}{\cos x}$

$= \int t^2 \cos^5 x \cdot \dfrac{dt}{\cos x} = \int t^2 (1-\sin^2 x)^2 \, dt$

$= \int t^2 (1-t^2)^2 dt$

$= \int (t^2 - 2t^4 + t^6)\, dt$

$$= \frac{t^3}{3} - \frac{2}{5}t^5 + \frac{t^7}{7} + c$$

$$= \frac{\sin^3 x}{3} - \frac{2}{5}\sin^5 x + \frac{\sin^7 x}{7} + c$$

(3) $\int \frac{\sin^4 x}{\cos^8 x} \cdot dx$

$$= \int \frac{\frac{\sin^4 x}{\cos^4 x}}{\frac{\cos^8 x}{\cos^4 x}} \, dx$$ [Devide numerator & denominator by $\cos^4 x$]

$$= \int \tan^4 x \cdot \sec^4 x \cdot dx$$

$$= \int \tan^4 x \, (1+\tan^2 x) \sec^2 x \cdot dx$$

$$= \int \tan^4 x \, (1+\tan^2 x) \sec^2 x \cdot dx$$

$\tan x = t \Rightarrow \sec^2 x \cdot dx = dt$

$$= \int t^4 (1+t^2) \, dt$$

$$= \frac{t^5}{5} + \frac{t^7}{7} + c$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$$

Evaluation of Integrals By Using Trigonometric Substitution :—

Expression	Substitution
a^2+x^2	$x = a\tan\theta$ or $a\cot\theta$
a^2-x^2	$x = a\sin\theta$ or $a\cos\theta$
x^2-a^2	$x = a\sec\theta$ or $a\operatorname{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$ or, $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(x-\beta)}$	$x = a\cos^2\theta + \beta\sin^2\theta$

Ex: $\int \frac{1}{(a^2-x^2)^{3/2}} \cdot dx$

Let $x = a\sin\theta$

$\Rightarrow dx = d(a\sin\theta)$

$\Rightarrow dx = a\cos\theta \cdot d\theta$

$$= \int \frac{1}{(a^2-a^2\sin^2\theta)^{3/2}}\, a\cos\theta \cdot d\theta$$

$$= \int \frac{a\cos\theta}{a^3\cos^3\theta} \cdot d\theta$$

$$= \frac{1}{a^2}\int \sec^2\theta \cdot d\theta$$

$$= \frac{1}{a^2}\tan\theta + C = \frac{1}{a^2}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{x}{a^3\left(\sqrt{1-\frac{x^2}{a^2}}\right)} + C$$

$$= \frac{x}{a^2\sqrt{a^2-x^2}} + C$$

(2) $\int \frac{x^2}{\sqrt{1-x}} \cdot dx$

$= \int \frac{x^2}{\sqrt{1-(\sqrt{x})^2}} \cdot dx$

Let $\sqrt{x} = \sin\theta$

$\Rightarrow x = \sin^2\theta$

$dx = d(\sin^2\theta)$

$= 2\sin\theta \cdot \cos\theta \cdot d\theta$

$= \int \frac{(\sin^2\theta)^2}{\sqrt{1-\sin^2\theta}} \cdot 2\sin\theta \cdot \cos\theta \cdot d\theta$

$= 2\int \sin^5\theta \cdot d\theta$

$= 2\int (1-\cos^2\theta)^2 \sin\theta \cdot d\theta$

Let $\cos\theta = u$

$\Rightarrow -\sin\theta \cdot d\theta = du$

$= -2\int (1-u^2)^2 du = -2\int (1-2u^2+u^4) du$

$= -2\left(u - \frac{2}{3}u^3 + \frac{u^5}{5}\right) + C$

$= \frac{-2}{15} u(15 - 10u^2 + 3u^4) + C$

$= \frac{-2}{15}(15 - 10\cos^2\theta + 3\cos^4\theta)\cos\theta + C$

$= \frac{-2}{15}\left\{15 - 10(1-\sin^2\theta) + 3(1-\sin^2\theta)^2\right\}\sqrt{1-\sin^2\theta} + C$

$= \frac{-2}{15}\left\{8 + 4\sin^2\theta + 3\sin^4\theta\right\}\sqrt{1-\sin^2\theta} + C$

$= \frac{-2}{15}\left\{8 + 4\sin^2\theta + 3\sin^4\theta\right\}\sqrt{1-\sin^2\theta} + C$

$= \frac{-2}{15}(8 + 4x + 3x^2)\sqrt{1-x} + C$

- **Theorem :-**

(i) $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

(ii) $\int \frac{1}{x^2-a^2} \cdot dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

(iii) $\int \frac{1}{a^2-x^2} \cdot dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

(iv) $\int \frac{1}{\sqrt{a^2-x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

(v) $\int \frac{1}{\sqrt{a^2+x^2}} \cdot dx = \log\left|x+\sqrt{a^2+x^2}\right| + c$

(vi) $\int \frac{1}{\sqrt{x^2-a^2}} \cdot dx = \log\left|x+\sqrt{x^2-a^2}\right| + c$

Ex: (1) $\int \frac{1}{4+9x^2}\, dx$

$= \frac{1}{9}\int \frac{1}{\frac{4}{9}+x^2} \cdot dx$

$= \frac{1}{9}\int \frac{1}{\left(\frac{2}{3}\right)^2+x^2} \cdot dx$

$= \frac{1}{9} \times \frac{1}{(2/3)} \tan^{-1}\left(\frac{x}{2/3}\right) + c$

$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + c$

(2) $\int \frac{1}{9x^2-4} \cdot dx$

$= \frac{1}{9}\int \frac{1}{x^2-\left(\frac{2}{3}\right)^2}\, dx = \frac{1}{9} \times \frac{1}{2\times\frac{2}{3}} \log\left|\frac{x-\frac{2}{3}}{x+\frac{2}{3}}\right| = \frac{1}{12}\log\left|\frac{3x-2}{3x+2}\right| + c$

③ $\int \frac{1}{16-9x^2}\cdot dx$

$= \frac{1}{9}\int \frac{1}{\frac{16}{9}-x^2}\cdot dx$

$= \frac{1}{9}\int \frac{1}{\left(\frac{4}{3}\right)^2-(x)^2}\, dx = \frac{1}{9}\times\frac{1}{2\times\frac{4}{3}}\times \log\left|\frac{\frac{4}{3}+x}{\frac{4}{3}-x}\right| + C$

$= \frac{1}{24}\log\left|\frac{4+3x}{4-3x}\right| + C$

④ $\int \frac{1}{\sqrt{9-25x^2}}\cdot dx$

$= \frac{1}{5}\int \frac{1}{\sqrt{\frac{9}{25}-x^2}}\cdot dx$

$= \frac{1}{5}\int \frac{1}{\sqrt{\left(\frac{3}{5}\right)^2-x^2}}\cdot dx$

$= \frac{1}{5}\sin^{-1}\left(\frac{x}{3/5}\right)+C = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right)+C$

⑤ $\int \frac{1}{x^2-x+1}\cdot dx$

$= \int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+1}\, dx$

$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\cdot dx$

$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\, dx$

$= \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{x - 1/2}{\sqrt{3}/2}\right) + c$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

(6) $\int \frac{1}{4x^2 - 4x + 3} \cdot dx$

$= \frac{1}{4} \int \frac{1}{x^2 - x + 3/4} \cdot dx$

$= \frac{1}{4} \int \frac{1}{x^2 - x + \frac{1}{4} - \frac{1}{4} + \frac{3}{4}} \cdot dx$

$= \frac{1}{4} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \cdot dx$

$= \frac{1}{4} \times \frac{1}{(1/\sqrt{2})} \tan^{-1}\left(\frac{x - 1/2}{1/\sqrt{2}}\right) + C$

$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$

(7) $\int \frac{1}{\sqrt{(x-1)(x-2)}} \cdot dx$

$= \int \frac{1}{\sqrt{x^2 - 3x + 2}} \cdot dx$

$= \int \frac{1}{\sqrt{x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2}} \cdot dx$

$= \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} \cdot dx$

$= \log\left|\left(x - \frac{3}{2}\right) + \sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C$

$= \log\left|\left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2}\right| + C$

(8) $\int \sqrt{\sec x - 1} \cdot dx$

$= \int \sqrt{\frac{1-\cos x}{\cos x}} \cdot dx$

$= \int \sqrt{\frac{1-\cos x}{\cos x} \times \frac{(1+\cos x)}{(1+\cos x)}} \cdot dx$

$= \int \sqrt{\frac{1-\cos^2 x}{\cos x + \cos^2 x}} \cdot dx$

$= \int \frac{\sin x}{\sqrt{\cos^2 x + \cos x}}$

Let $\cos x = t$

$\Rightarrow -\sin x \cdot dx = dt$

$\Rightarrow dx = \frac{-dt}{\sin x}$

$= \int \frac{-dt}{\sqrt{t^2+t}} = -\int \frac{dt}{\left(t+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$

$= -\log\left|\left(t+\frac{1}{2}\right) + \sqrt{\left(t+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} + C\right.$

$= -\log\left|\left(t+\frac{1}{2}\right) + \sqrt{t^2+t}\right| + C$

$= -\log\left|\left(\cos x + \frac{1}{2}\right) + \sqrt{\cos^2 x + \cos x}\right| + C$

(9) $\int \frac{x+5}{\sqrt{x^2+6x-7}} \cdot dx$

$= \int \frac{x+3+2}{\sqrt{(x+3)^2-16}} \cdot dx$

$= \int \frac{z+2}{\sqrt{z^2-16}} dz$, putting $x+3 = z$
$\Rightarrow dx = dz$

$$= \int \frac{z}{\sqrt{z^2-16}}\,dz + 2\int \frac{dz}{\sqrt{z^2-16}}$$

$$= \frac{1}{2}\int \frac{d(z^2-16)}{\sqrt{z^2-16}} + 2\int \frac{dz}{\sqrt{z^2-16}}$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{u}} + 2\int \frac{dz}{\sqrt{z^2-16}}$$

$$= \sqrt{u} + 2\ln\left|z + \sqrt{z^2-16}\right| + C$$

$$= \sqrt{z^2-16} + 2\ln\left|z + \sqrt{z^2-16}\right| + C$$

$$= \sqrt{(x+3)^2-16} + 2\ln\left|x+3+\sqrt{(x+3)^2-16}\right| + C$$

$$= \sqrt{x^2+6x-7} + 2\ln\left|(x+3) + \sqrt{x^2+6x-7}\right| + C$$

INTEGRATION BY PARTS :-

If u & v are two functions of x, then

$$\int uv \cdot dx = u\int v \cdot dx - \int\left\{\frac{du}{dx}\int v \cdot dx\right\} dx$$

i.e The Integral of the product of of two functions =

(first function) × (Integral of Second function)

− Integral of {(Differentiation of first function) × (Integral of Second function)}

ILATE Rule :—

The first function can be choose using ILATE Rule where I = Inverse Trigonometric fun

L = Logarithmic fun'

A = Algebraic function

T = Trigonometric function

E = Exponential function

Problems :—

(1) $\int \underset{\mathrm{I}}{x} \underset{\mathrm{II}}{\sin 3x} \cdot dx$

$$= x\left\{\int \sin 3x \cdot dx\right\} - \int\left\{\frac{d}{dx}(x) \times \int \sin 3x \cdot dx\right\} dx$$

$$= x \times \frac{-1}{3}\cos 3x - \int\left\{\frac{-1}{3}\cos 3x\right\} \cdot dx$$

$$= \frac{-1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x \cdot dx$$

$$= \frac{-1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$$

(2) $\int x \log x \, dx$

$= \log x \left\{\int x \cdot dx\right\} - \int \left\{\frac{d}{dx}(\log x) \times \int x \cdot dx\right\} dx$

$= (\log x)\frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2} \cdot dx$

$= \frac{x^2}{2}\log x - \frac{1}{2}\int x \cdot dx$

$= \frac{x^2}{2}\log x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C$

$= \frac{x^2}{2}\log x - \frac{1}{4}x^2 + C$

(3) $\int \log x \cdot dx$

$= \int \underset{\mathrm{I}}{\log x} \cdot \underset{\mathrm{II}}{1} \cdot dx$

$= \log x \left\{\int 1 \cdot dx - \int \frac{1}{\cancel{x}} \cdot x \cdot dx\right.$

$= x\log x - x + C$

(4) $\int e^x(\sin x + \cos x)\, dx$

$= \int \underset{\mathrm{II}}{e^x} \underset{\mathrm{I}}{\sin x} \cdot dx + \int e^x \cos x \cdot dx$

$= (\sin x)e^x - \cancel{\int \cos x \cdot e^x \cdot dx} + \int e^x \cos x \cdot dx + C$

$= e^x \sin x + C$

(5) $\int \sin^{-1}x . dx$

Let $\cancel{\int} \sin^{-1}x . \cancel{dx} = t$

Then $x = \sin t$

$\Rightarrow dx = \cos t \cdot dt$

$$I = \int \underset{\mathrm{I}}{t} \underset{\mathrm{II}}{\cos t} \cdot dt$$

$$= t\sin t - \int 1 . \sin t \cdot dt$$

$$= t\sin t - \int \sin t \cdot dt$$

$$= t\sin t + \cos t + C$$

$$= x\sin^{-1}x + \sqrt{1-\sin^2 t} + C$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

(6) $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

$$= \int e^x \cdot \frac{1}{x} . dx - \int e^x . \frac{1}{x^2} . dx$$

$$= \frac{1}{x}e^x - \int \frac{-1}{x^2} e^x . dx - \int e^x \times \frac{1}{x^2} . dx + C$$

$$= \frac{1}{x}e^x + \cancel{\int \frac{1}{x^2} e^x . dx} - \cancel{\int \frac{1}{x^2} e^x . dx} + C$$

$$= \frac{1}{x}e^x + C$$

SOME IMPORTANT INTEGRALS :—

Theorem

(i) $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$

(ii) $\int \sqrt{a^2 + x^2}\, dx = \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{1}{2} a^2 \log\left|x + \sqrt{a^2 + x^2}\right| + c$

(iii) $\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{1}{2} a^2 \log\left|x + \sqrt{x^2 - a^2}\right| + c$

Problems:

(1) $\int \sqrt{4x^2 + 9}\, dx$

$= 2 \int \sqrt{x^2 + \frac{9}{4}}\, dx$

$= 2 \int \sqrt{x^2 + \left(\frac{3}{2}\right)^2}\, dx$

$= 2 \left\{ \frac{1}{2} x \sqrt{x^2 + \frac{9}{4}} + \frac{1}{2}\left(\frac{3}{2}\right)^2 \log\left| x + \sqrt{x^2 + \frac{9}{4}} \right| \right\} + c$

$= \frac{x}{2} \sqrt{4x^2 + 9} + \frac{9}{4} \log\left|2x + \sqrt{4x^2 + 9}\right| + c$

(2) $\int \frac{x^2}{\sqrt{1 - 2x - x^2}}\, dx$

$= -\int \frac{-x^2}{\sqrt{1 - 2x - x^2}}\, dx = -\int \frac{(1 - 2x - x^2) + (2x - 1)}{\sqrt{1 - 2x - x^2}}\, dx$

$= -\int \sqrt{1 - 2x - x^2}\, dx - \int \frac{2x - 1}{\sqrt{1 - 2x - x^2}}\, dx$

$= -\int \sqrt{1 - 2x - x^2}\, dx + \int \frac{-2x - 2 + 3}{\sqrt{1 - 2x - x^2}}\, dx$

$$= -\int\sqrt{1-2x-x^2}\,dx + \int\frac{-2x-2}{\sqrt{1-2x-x^2}}\,dx + 3\int\frac{1}{\sqrt{1-2x-x^2}}\,dx$$

$$= -\int\sqrt{(\sqrt{2})^2-(x+1)^2}\,dx + \int\frac{1}{\sqrt{1-2x-x^2}}\,d(1-2x-x^2)$$
$$+ 3\int\frac{1}{\sqrt{(\sqrt{2})^2-(x+1)^2}}\,dx$$

$$= \frac{-1}{2}\left\{(x+1)\sqrt{1-2x-x^2} + 2\sin^{-1}\frac{x+1}{\sqrt{2}}\right\} + 2\sqrt{1-2x-x^2} + 3\sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$
$$+C$$

(3) $\int(x-5)\sqrt{x^2+x}\,dx$

let $x-5 = \lambda\frac{d}{dx}(x^2+x)+\mu$

$\Rightarrow x-5 = \lambda(2x+1)+\mu$

comparing coefficients of like powers of x, we get,

$1 = 2\lambda$ & $\lambda+\mu = -5 \Rightarrow \lambda = \frac{1}{2}$ & $\mu = -\frac{11}{2}$

$$= \int(x-5)\sqrt{x^2+x}\,dx$$

$$= \int\left(\frac{1}{2}(2x+1)-\frac{11}{2}\right)\sqrt{x^2+x}\,dx$$

$$= \int\frac{1}{2}(2x+1)\sqrt{x^2+x}\,dx - \frac{11}{2}\int\sqrt{x^2+x}\,dx$$

$$= \frac{1}{2}\int\sqrt{t}\,dt - \frac{11}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx \quad \text{where } t = x^2+x$$

$$= \frac{1}{2}\times\frac{t^{3/2}}{3/2} - \frac{11}{2}\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right.$$
$$\left. - \frac{1}{2}\left(\frac{1}{2}\right)^2\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|\right\} + C$$

$$= \frac{1}{3} t^{3/2} - \frac{11}{2}\left\{ \frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right\} + c$$

$$= \frac{1}{3}\left(x^2+x\right)^{3/2} - \frac{11}{2}\left\{ \frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\log\left|\left(x+\frac{1}{2}\right)\sqrt{x^2+x}\right|\right\}$$

INTEGRATION OF RATIONAL ALGEBRAIC FUNCTIONS BY USING PARTIAL FRACTIONS ;—

if $f(x)$ & $g(x)$ are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function or a rational function of x.

(i) if degree of $f(x) <$ degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called a proper rational function.

(ii) if degree of $f(x) \geq$ degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called Improper rational function.

– For improper rational function, we divide $f(x)$ by $g(x)$ & expressed as

$$\frac{f(x)}{g(x)} = \phi(x) + \frac{\psi(x)}{g(x)}$$

Where $\phi(x)$ & $\psi(x)$ are polynomials such that the degree of $\psi(x)$ is less than that of $g(x)$., then it reduce to proper rational function.

→ For proper fraction $\frac{f(x)}{g(x)}$ can be decomposed into simpler fractions, called partial fractions & each simpler fractions integrated separately.

→ 4 types of different cases arises depending on denominator $Q(x)$.

(i) if $g(x)$ is non-repeating factor, i.e

$$g(x) = (a_1x+b_1)(a_2x+b_2)(a_3x+b_3)$$

$$\therefore \frac{f(x)}{g(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \frac{A_3}{a_3x+b_3}$$

(ii) if $g(x)$ has some repeating factor,

let, $g(x) = (ax+b)^3(cx+d)$

$$\therefore \frac{f(x)}{g(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \frac{A_4}{(cx+d)}$$

(iii) for a non-repeated quadratic factor lx^2+px+q,

if $g(x) = (lx^2+px+q)(ax+b)$

$$\therefore \frac{f(x)}{g(x)} = \frac{A_1x+B_1}{lx^2+px+q} + \frac{A_2}{ax+b}$$

(iv) For repeated quadratic factor $(lx^2+px+q)^n$

i.e if $g(x) = (lx^2+px+q)^2(ax+b)$

$$\frac{f(x)}{g(x)} = \frac{A_1x+B_1}{lx^2+px+q} + \frac{A_2x+B_2}{(lx^2+px+q)^2} + \frac{A_3}{ax+b}.$$

Problems :-

Ex-1: $\int \frac{4x+5}{x^2+x-2} \cdot dx$

$$= \int \frac{4x+5}{(x+2)(x-1)} \cdot dx \quad \text{———— (1)}$$

$$\frac{4x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \quad \text{———— (2)}$$

$$\frac{4x+5}{(x+2)(x-1)} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$$\Rightarrow \frac{4x+5}{(x+2)(x-1)} = \frac{Ax - A + Bx + 2B}{(x+2)(x-1)}$$

$$\Rightarrow \frac{4x+5}{\cancel{(x+2)(x-1)}} = \frac{(A+B)x - A + 2B}{\cancel{(x+2)(x-1)}}$$

$$\Rightarrow 4x+5 = (A+B)x - A - 2B$$

Now comparing coefficients of x,

$$A + B = 4$$
$$A + 2B = 5$$

$$3B = 9 \quad\Big|\quad A = 4 - B$$
$$\Rightarrow B = 3 \quad\Big|\quad A = 4-3$$
$$\quad\Big|\quad A = 1$$

Now putting the value of A & B in eq^n (2) :-

$$\frac{4x+5}{(x+2)(x-1)} = \frac{1}{x+2} + \frac{3}{x-1}$$

Now Integrating ~~on both sides~~ with respect to x,

$$= \int \frac{4x+5}{(x+2)(x-1)} \cdot dx = \int \left(\frac{1}{x+2} + \frac{3}{x-1} \right) \cdot dx$$

$$= \int \frac{1}{x+2} \cdot dx + \int \frac{3}{x-1} \cdot dx$$

$$= \ln|x+2| + 3\ln|x-1| + C$$

Ex-2: - $\int \frac{x^2}{(x+1)^2(x-2)} \cdot dx$

$$\frac{x^2}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

$$\Rightarrow \frac{x^2}{\cancel{(x+1)^2(x-2)}} = \frac{A(x+1)(x-2) + B(x-2) + C(x+1)^2}{(x+1)^2(x-2)}$$

$$= \frac{A(x^2 - 2x + x - 2) + B(x-2) + C(x^2 + 2x + 1)}{\cancel{(x+1)^2(x-2)}}$$

$$\Rightarrow x^2 = (A+C)x^2 - (A - B - 2C)x - 2C - 2B + C$$

Now comparing the coefficient of x,

$$\Rightarrow B = \frac{-1}{3}$$

$$C = \frac{4}{9}$$

$$A = \frac{5}{9}$$

$$\therefore \frac{x^2}{(x+1)^2(x-2)} = \frac{\frac{5}{9}}{x+1} - \frac{\frac{1}{3}}{(x+1)^2} + \frac{\frac{4}{9}}{x-2}$$

$$= \frac{5}{9} \times \frac{1}{x+1} - \frac{1}{3} \frac{1}{(x+1)^2} + \frac{4}{9} \frac{1}{(x-2)}$$

Now, Integrating on both sides w.r.to x.

$$\Rightarrow \int \frac{x^2}{(x+1)^2(x-2)} \cdot dx = \frac{5}{9}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{dx}{(x+1)^2} + \frac{4}{9}\int \frac{dx}{x-2}$$

$$= \frac{5}{9}\ln|x+1| - \frac{1}{3}\times\frac{1}{x+1} + \frac{4}{9}\ln|x-2| + C$$

(iii) $\int \frac{2x^2+x+3}{(x^2+2)(x-1)} \cdot dx$

$$\frac{2x^2+x+3}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1}$$

$$\therefore \quad 2x^2+x+3 = (Ax+B)(x-1) + C(x^2+2)$$

Now comparing the coeffecient of x^2, x..

$A=0, B=1, C=2$

$$\therefore \quad \frac{2x^2+x+3}{(x^2+2)(x-1)} = \frac{1}{x^2+2} + \frac{2}{x-1}$$

Now Integrating on both side with respect to x, we get

$$\int \frac{2x^2+x+3}{(x^2+2)(x-1)} \cdot dx = \int \frac{dx}{x^2+2} + 2\int \frac{dx}{x-1}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + 2\ln|x-1| + C$$

~~Easy~~:-

(vii) Integral of the form

$$\int \frac{x^2+1}{x^4+\lambda x^2+1}\,dx, \quad \int \frac{x^2-1}{x^4+\lambda x^2+1}\,dx, \quad \int \frac{1}{x^4\pm\lambda x^2+1}\,dx,$$

<u>Algorithm</u>

<u>Step-I</u>: Divide numerator & denominator by x^2

<u>Step-II</u>: Express the denominator of integrand in the form

$$\left(x+\frac{1}{x}\right)^2 \pm k^2$$

<u>Step-III</u>: Introduce $d\left(x+\frac{1}{x}\right)$ or $d\left(x-\frac{1}{x}\right)$ or both in the numerator.

<u>Step-IV</u> Substitute $x+\frac{1}{x}=t$ or $x-\frac{1}{x}=t$ as the case may be. This substitution reduces the integrals in one of the following forms $\int \frac{1}{x^2\pm a^2}\,dx$, $\int \frac{1}{x^2-a^2}\,dx$.

<u>Step-V</u>: Use the appropriate formula.

<u>Problems</u>: —

$$\int \frac{x^2-1}{x^4+x^2-1}\,dx$$

$$= \int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx$$

$$= \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-(1)^2}$$

Let $x+\frac{1}{x}=u$

$\Rightarrow d\left(x+\frac{1}{x}\right)=du$

$\Rightarrow \left(1-\frac{1}{x^2}\right)dx=du$

$$= \int \frac{du}{(u^2-1)^2} = \frac{1}{2(1)}\log\left|\frac{u-1}{u+1}\right|+c$$

$$= \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right|+C$$

$$= \frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right|+C$$

INTERGRATION OF SOME SPECIAL IRRATIONAL ALGEBRAIC FUNCTION :-

(I) Integral of the form $\int \frac{dx}{(ax+b)\sqrt{cx+d}}$ & $\int \frac{1}{(ax^2+bx+c)\sqrt{px+q}}$

Ex:(i) $\int \frac{1}{(x-3)\sqrt{x+1}}\,dx$

let $x+1 = t^2 \Rightarrow dx = 2t\cdot dt$

$= \int \frac{1}{(t^2-1-3)} \times \frac{2t}{\sqrt{t^2}}\,dt$

$= 2\int \frac{dt}{t^2-2^2} = 2\times\frac{1}{2(2)}\log\left|\frac{t-2}{t+2}\right| + C$

$= \frac{1}{2}\log\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right| + C$

(II) $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$

let $x+1 = t^2$

$\Rightarrow dx = 2t\cdot dt$

$= \int \frac{(t^2+2)2t}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}\,dt$

$= 2\int \frac{(t^2+1)}{t^4+t^2+1}\,dt = 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\,dt$

$= 2\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{3})^2}\,dt = 2\int \frac{du}{u^2+(\sqrt{3})^2}$, where $t-\frac{1}{t}=u$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left\{\frac{t - \frac{1}{t}}{\sqrt{3}}\right\} + C$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t^2-1}{\sqrt{3}\, t}\right) + C$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left\{\frac{x}{\sqrt{3(x+1)}}\right\} + C$

Integral of the form $\int \frac{1}{(ax+b)\sqrt{px^2+qx+r}} \cdot dx$:—

Put $ax+b = \frac{1}{t}$

Ex! $\int \frac{1}{(x+1)\sqrt{x^2-1}} \cdot dx$

let $x+1 = \frac{1}{t}$

$\Rightarrow dx = \frac{-1}{t^2} \cdot dt$

$= \int \frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2 - 1}} \times \left(\frac{-1}{t^2}\right) \cdot dt$

$= -\int \frac{dt}{\sqrt{1-2t}} = -\int (1-2t)^{-1/2} \cdot dt = \frac{-(1-2t)^{1/2}}{(-2)\left(\frac{1}{2}\right)} + C$

$= \sqrt{1-2t} + C$

$= \sqrt{1 - \frac{2}{x+1}} + C$

$= \sqrt{\frac{x-1}{x+1}} + C$

Integration of the form $\int \frac{1}{(ax^2+b)\sqrt{cx^2+d}} \cdot dx$, —

Here put $x = \frac{1}{t}$

Ex:- $\int \frac{1}{x^2(\sqrt{1+x^2})} dx$

let $x = \frac{1}{t}$ & $\frac{-1}{x^2} \cdot dx = dt$

$\Rightarrow dx = -x^2 \cdot dt$

$= \int \frac{-dt}{\sqrt{1+\frac{1}{t^2}}}$

$= -\int \frac{t \cdot dt}{\sqrt{t^2+1}} = -\int \frac{u \cdot du}{\sqrt{u^2}}$, where $t^2+1 = u^2$

$= \int -1 \cdot du = -u + c = -\sqrt{t^2+1} + C = -\sqrt{\frac{1}{x^2}+1} + C$

$= -\frac{\sqrt{1+x^2}}{x} + C$

Trigonometric fun' of the form $\int \frac{dx}{a+b\cos x + c\sin x}$: —

This can be evaluated by converting $\cos x$ & $\sin x$ to $\tan\frac{x}{2}$ $(= t)$

Ex:- $\int \frac{dx}{2+\sin x}$

$= \int \frac{dx}{2 + \frac{2\tan x/2}{1+\tan^2 x/2}}$

$$= \frac{1}{2}\int \frac{1+\tan^2 x/2}{\tan^2 x/2 + 1 + \tan x/2}\, dx$$

$$= \frac{1}{2}\int \frac{\sec^2 x/2}{\tan^2 x/2 + \tan x/2 + 1}\, dx$$

$$= \int \frac{dt}{t^2+t+1} \quad \therefore \text{ Where } \tan x/2 = t$$

$$= \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan x/2 + 1}{\sqrt{3}}\right) + c$$

DEFINITE INTEGRAL: —

Fundamental Theorem of Integral Calculus: —

statement :- Let $\phi(x)$ be the primitive or antiderivative of a continuous function $f(x)$ defined on $[a,b]$ i.e $\frac{d}{dx}[\phi(x)] = f(x)$. Then the definite Integral of $f(x)$ over $[a,b)$ is denoted by $\int_a^b f(x).dx$ & is equal to $[\phi(b) - \phi(a)]$

i.e; $\int_a^b f(x).dx = \phi(b) - \phi(a)$.

Where $a \rightarrow$ Lower Limit

$b \rightarrow$ Upper Limit

$[a,b] \rightarrow$ Interval of Integration

Evaluation of Definite Integral :-

Algorithm :-

Step-I :- Find the Indefinite integral $\int f(x) \cdot dx$

let this be $\phi(x)$.

There is no need to Keep constant of integration.

Step-II :- Evaluate $\phi(b)$ & $\phi(a)$

Step-III :- Calculate $\phi(b) - \phi(a)$ & this will be the answer.

Q(1) - $\int_1^2 x^2 \cdot dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

(2) $\int_0^1 \frac{1}{2x-3} \cdot dx = \left[\frac{1}{2}\log(2x-3)\right]_0^1$

$= \frac{1}{3}\left[\log|-1| - \log|-3|\right]$

$= \frac{1}{2}(\log 1 - \log 3)$

$= \frac{1}{2}(0 - \log 3) = -\frac{\log 3}{2}$

Elementary properties of Definite Integrals :-

(1) $\int_a^b f(x).dx = -\int_b^a f(x).dx$

(2) $\int_a^b f(x).dx = \int_a^b f(y).dy = \int_a^b f(z).dz$

(3) $\int_a^b f(x).dx = \int_a^\alpha f(x).dx + \int_\alpha^b f(x).dx$, where $a < \alpha < b$

(4) $\int_0^a f(x).dx = \int_0^a f(a-x).dx$

(5) $\int_{-a}^a f(x).dx = \begin{cases} 2\int_0^a f(x).dx & \text{if } f \text{ is even function} \\ 0 & \text{, if } f \text{ is odd function.} \end{cases}$

(6) $\int_0^{2a} f(x).dx = \begin{cases} 2\int_0^a f(x).dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$.

Ex:- (1) $\int_1^4 [x].dx = \int_1^2 [x].dx + \int_2^3 [x].dx + \int_3^4 [x].dx$

$= \int_1^2 1.dx + \int_2^3 2.dx + \int_3^4 3.dx$

$= 2-1+2(3-2)+3(4-3) = 6$.

(11) $\int_{-3}^4 |x|.dx = \int_{-3}^0 |x|.dx + \int_0^4 |x|.dx$

$= \int_{-3}^0 (-x)dx + \int_0^4 x.dx = \int_0^{-3} x.dx + \int_0^4 x.dx$

$= \left|\frac{x^2}{2}\right|_0^{-3} + \left|\frac{x^2}{2}\right|_0^4 = \frac{9}{2} - 0 + \frac{16}{2} - 0 = \frac{25}{2}$

(3) $\int_0^{\frac{\pi}{2}} \ln \sin x \cdot dx$

$$I = \int_0^{\pi/2} \ln \sin\left(\pi/2 - x\right) \cdot dx = \int_0^{\pi/2} \ln \cos x \cdot dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \ln \cos x \cdot dx + \int_0^{\pi/2} \ln \sin x \cdot dx$$

$$= \int_0^{\pi/2} \left(\ln \cos x + \ln \sin x\right) dx$$

$$= \int_0^{\pi/2} \ln\left(\cos x \times \sin x\right) \cdot dx$$

$$= \int_0^{\pi/2} \ln\left(\frac{\sin 2x}{2}\right) \cdot dx$$

$$2I = \int_0^{\pi/2} \ln \sin 2x \cdot dx - \int_0^{\pi/2} \ln 2 \cdot dx \quad \text{———} (1)$$

Now $\int_0^{\pi/2} \ln \sin 2x \cdot dx$

$$= \frac{1}{2} \int_0^{\pi} \ln \sin z \cdot dz, \quad \text{putting } z = 2x$$

$$= \int_0^{\pi/2} \ln \sin x \cdot dx = I$$

Now from eqⁿ (1)

$$\Rightarrow 2I = I - \int_0^{\pi/2} \ln 2 \cdot dx$$

$$= I - \pi/2 \ln 2$$

$$\Rightarrow I = -\pi/2 \ln 2 = \pi/2 \ln\left(\frac{1}{2}\right)$$

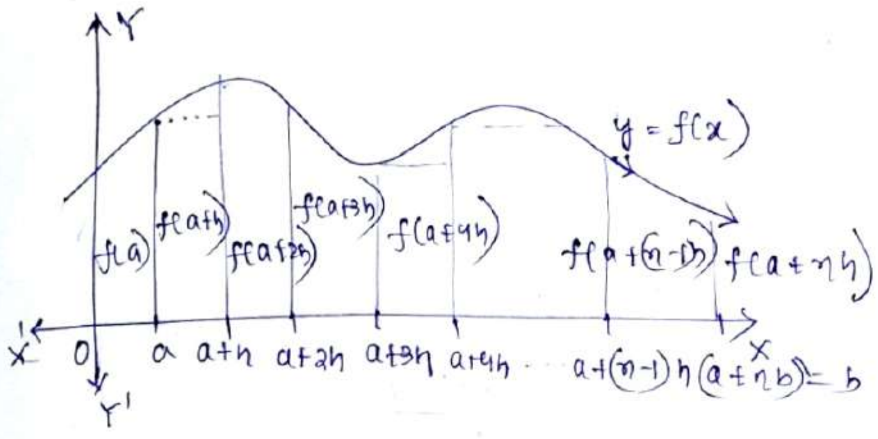
Area under plane curves :-

Integration As the Limit of A Sum :-

Let $f(x)$ be a continuous real valued function defined on the closed interval $[a,b]$ which is divided into n equal parts each of ~~which~~ width h by inserting $(n-1)$ points $a+h, a+2h, a+3h, \ldots a+(n-1)h$ between a & b, then

$nh = b-a$

$\Rightarrow h = \frac{b-a}{n}$



Let S_n = Sum of Areas of n rectangles

$\Rightarrow S_n = h \cdot f(a) + h f(a+h) + h \cdot f(a+2h) + \ldots + h \cdot f(a+(n-1)h)$

$= h[f(a) + f(a+h) + f(a+2h) + \cdots f(a+(n-1)h)]$

Now as $n \to \infty$, S_n gives area of the region bounded by the curve $y = f(x)$, $y = 0$ (x-axis) & ordinates $x = a$ & $x = b$.

$\Rightarrow$ This $\lim_{n\to\infty} S_n$ exist for all continuous functions defined on closed integral $[a,b]$ which is the definite integral of $f(x)$ over $[a,b]$.

$$\therefore \lim_{n \to \infty} S_n = \int_a^b f(x).dx$$

or $$\int_a^b f(x).dx = \lim_{n \to \infty} h\left[f(a) + f(a+h) + f(a+2h) + \cdots f(a+(n-1)h)\right]$$

where $h = \dfrac{b-a}{n}$

or $$\lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \cdots f(a+(n-1)h)\right]$$

$[\because n \to \infty \Leftrightarrow h \to 0]$

→ This method is also called Integration by ab-initio method or integration as the limit of a sum;

Note:-

1. $1 + 2 + 3 + \cdots + (n-1) = \dfrac{n(n-1)}{2}$
2. $1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 = \dfrac{n(n-1)(2n-1)}{6}$
3. $1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 = \left\{\dfrac{n(n-1)}{2}\right\}^2$
4. $a + ar + ar^2 + \cdots + ar^{n-1} = a\left(\dfrac{r^n - 1}{r-1}\right), r \neq 1$

Problems :—

① $\int_0^2 (x+4)\,dx$

Now we have

$$\int_a^b f(x).\,dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+\cdots f(a+(n-1)h)]$$

where $h = \frac{b-a}{n}$

Here $a=0, b=2, f(x)=x+4$ & $h=\frac{2-0}{n}=\frac{2}{n}$

$$\therefore \int_0^2 (x+4)\,dx = \lim_{h\to 0} h[f(0)+f(0+h)+f(0+2h)+\cdots f(0+(n-1)h)]$$

$$= \lim_{h\to 0} h[(0+4)+(h+4)+(2h+4)+\cdots+((n-1)h+4)]$$

$$= \lim_{h\to 0} h[4n + h(1+2+3+\cdots+(n-1))]$$

$$= \lim_{h\to 0} h\left\{4n + h\,\frac{n(n-1)}{2}\right\}$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[4n + \frac{2}{n}\times\frac{n(n-1)}{2}\right] \qquad \left[h=\frac{2}{n} \text{ \& } h\to 0 \Rightarrow n\to\infty\right]$$

$$= \lim_{n\to\infty}\left\{8+2\left(1-\frac{1}{n}\right)\right\} = 8+2(1-0) = 10$$

② $\int_0^2 e^x.\,dx$

We have, $\int_a^b f(x).\,dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h)]$

where $h=\frac{b-a}{n}$

Here $a = 0,\ b = 2,\ f(x) = e^x,\ h = \frac{2}{n}$

$$\int_0^2 e^x \cdot dx = \lim_{h \to 0} h[f(0) + f(h) + f(2h) + \cdots f(n-1)h]$$

$$= \lim_{h \to 0} h[e^0 + e^h + e^{2h} + \cdots e^{(n-1)h}]$$

$$= \lim_{h \to 0} h\left[e^0 \left\{\frac{(e^h)^n - 1}{e^h - 1}\right\}\right]$$

$$= \lim_{h \to 0} h \left\{\frac{e^{nh} - 1}{e^h - 1}\right\}$$

$$= \lim_{h \to 0} \frac{h}{h} \left\{\frac{e^2 - 1}{\left(\frac{e^h - 1}{h}\right)}\right\} \quad \left[h = \frac{2}{n} \Rightarrow nh = 2\right]$$

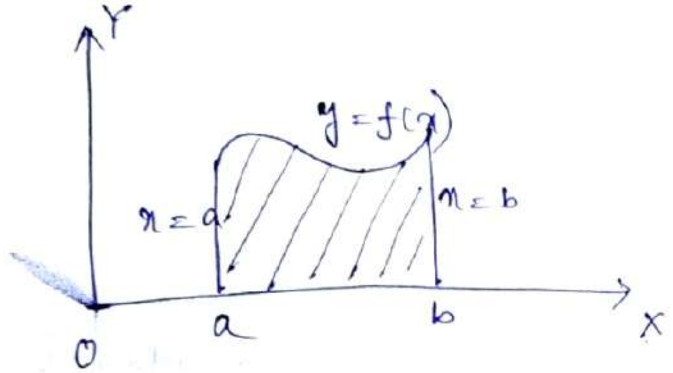
$$= \lim_{h \to 0} \frac{e^2 - 1}{\left(\frac{e^h - 1}{h}\right)} = \frac{e^2 - 1}{1} = e^2 - 1 \quad \left[\lim_{h \to 0} \frac{e^h - 1}{h} = 1\right]$$

Areas As a Definite Integral: —

Theorem: — Let $f(x)$ be a continuous function defined on $[a, b]$. Then the area bounded by the curve $y = f(x)$, the x-axis & the ordinates $x = a$ & $x = b$ is given by $\int_a^b f(x) \cdot dx$ or $\int_a^b y \cdot dx$.

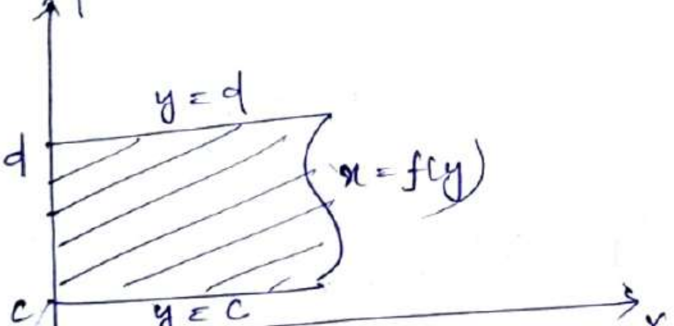
Def-1 :- Area of the region bounded by the curve $y = f(x)$ the X-axis & the lines $x = a$, $x = b$ is defined by ..

$$\text{Area} = \left| \int_a^b y \cdot dx \right| = \left| \int_a^b f(x) \cdot dx \right|$$



Def^n-2 :- Area of the region by the curve $x = f(y)$, the Y-axis & the lines $y = c$, $y = d$ is defined by

$$\text{Area} = \int_c^d x \cdot dy = \left| \int_c^d f(y) \cdot dy \right|$$



Problems :-

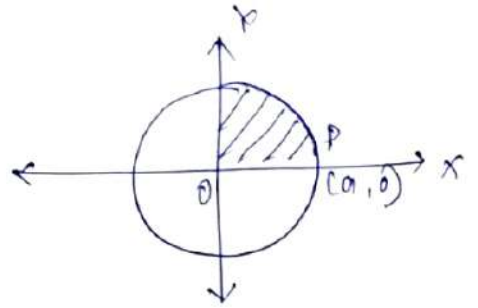
(1) Find the area of the region bounded by the curve $y = e^{3x}$, x-axis & the lines $x = 4$ & $x = 2$.

Ans: The required area is defined by.

$$A = \int_2^4 e^{3x}\cdot dx = \frac{1}{3}e^{3x}\Big]_2^4 = \frac{1}{3}\left(e^{12x} - e^{6x}\right)$$

Ex-2: Find the area of the circle $x^2+y^2=a^2$?

Sol^n: We observe that, $y=\sqrt{a^2-x^2}$ in the first quadrant



$\therefore$ The area of the circle in the first quadrant is defined by,

$$A_1 = \int_0^a \sqrt{a^2-x^2}\cdot dx$$

$$\text{Total Area } (A) = 4\int_0^a \sqrt{a^2-x^2}\cdot dx$$

$$= 4\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

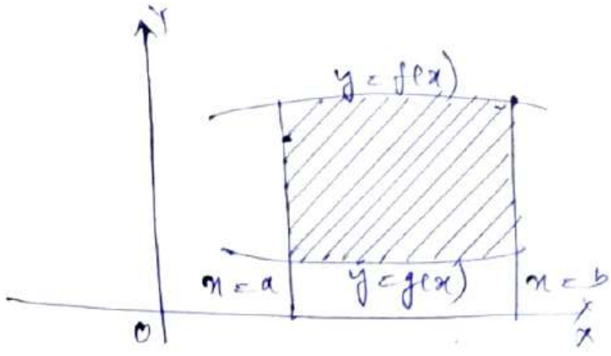
$$= 4\frac{a^2}{2}\sin^{-1}1 = 2a^2\frac{\pi}{2} = \pi a^2.$$

Area Between Two curves: —

If there two curves $y=f(x)$, $y=g(x)$ with $g(x) < f(x)$ in $[a,b]$, then the area between them & between the ordinates $x=a$ & $x=b$ is given by

$$A = \int_a^b f(x)\cdot dx - \int_a^b g(x)\, dx$$

$$= \int_a^b [f(x) - g(x)]\cdot dx$$

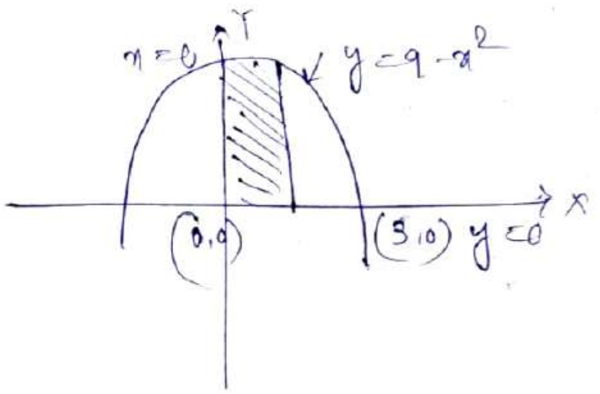


Ex 1:2) (i) Area of the region enclosed by $y = 9 - x^2$, $y = 0$ & the ordinates $x = 0$ & $x = 2$ is given by

$$A = \int_0^2 (9 - x^2)\, dx$$

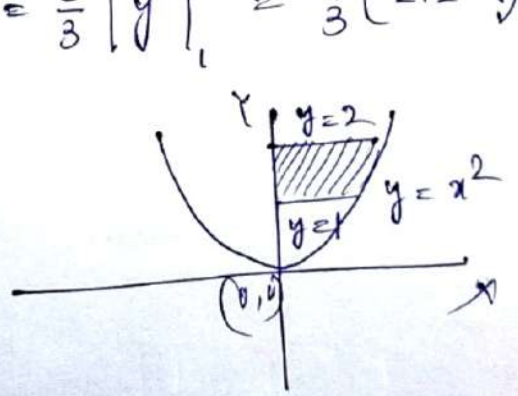
$$= \left[9x - \frac{x^3}{3}\right|_0^2$$

$$= 18 - \frac{8}{3} = \frac{46}{3}$$



(ii) The Area bounded by $x^2 = y$, the y-axis ($x = 0$) & the lines $y = 1$ & $y = 2$ is given by

$$A = \int_1^2 x\cdot dy = \int_1^2 \sqrt{y}\cdot dy = \frac{2}{3}\left|y^{\frac{3}{2}}\right|_1^2 = \frac{2}{3}\left(2\sqrt{2} - 1\right)$$



Assignments →

(1) Find the Integrals :-

(a) $y = \int 4x^3 \cdot dx$

(b) $\int x^6 \cdot dx$

(c) $\int \left(2\sqrt{x} + \frac{3}{\sqrt{x}}\right) dx$

(d) $\int \left(x^{4/3} + \frac{1}{x^{1/3}}\right) dx$

(e) $\int \frac{1 - \cos 2x}{1 + \cos 2x} \cdot dx$

(f) $\int \frac{a \sin^3 x + b \cos^3 x}{\sin^2 2x} \cdot dx$

(g) $\int 3^x \cdot dx$

(h) $\int \left(\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}\right) dx$