ENGINEERING MATH - III

3rd Semester

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FOURIER SERIES

Class-I

Periodic function:—

Defn:- A function $f(x)$ is said to be of period T or to be periodic with period $T > 0$ if for all real x, $f(x+T) = f(x)$ and T is the least of such values. In other words, if the value of each ordinate $f(x)$ repeats itself at equal intervals in the abscissa, $f(x)$ is said to be a periodic function.

- If T is the period of $f(x)$, then

$$f(x) = f(x+T) = f(x+2T) = \cdots = f(x+nT) = \cdots$$

Also $f(x) = f(x-T) = f(x-2T) = \cdots = f(x-nT) = \cdots$

$\therefore f(x) = f(x \pm nT)$, where n is a positive integer.

Thus $f(x)$ represents itself after period of T.

- For example $\sin x$, $\cos x$, $\sec x$ & $\operatorname{cosec} x$ are periodic functions with period 2π while $\tan x$ & $\cot x$ are periodic functions with period π. The function $\sin nx$ & $\cos nx$ are periodic with period $\frac{2\pi}{n}$.

- The sum of a no of periodic function is also periodic. If T_1 & T_2 are the periods of $f(x)$ & $g(x)$, then the period of $af(x) + bg(x)$ is the least common multiple of T_1 & T_2

- For example, $\cos x$, $\cos 2x$, $\cos 3x$ are periodic functions with periods 2π, π & $\frac{2\pi}{3}$.

Example - 1 :

- If f & g are periodic function with same period T, show that $(c_1 f + c_2 g)$ is also a periodic function of period T, where c_1 & c_2 are real numbers.

Solution :-

Since f & g are periodic function with period T.

we have, $f(x+T) = f(x)$

$$g(x+T) = g(x)$$

$$(c_1 f + c_2 g)(x+T) = c_1 f(x+T) + c_2 g(x+T)$$
$$= c_1 f(x) + c_2 g(x)$$
$$= (c_1 f + c_2 g)(x)$$

$\therefore (c_1 f + c_2 g)$ is periodic with period T.

Example - 2 :-

- Express the function $f(x)$ as sum of an even function & an odd function.

Solution :- $f(x)$ can be written as

$$f(x) = \frac{1}{2}\left[f(x) + f(x)\right]$$

$$= \frac{1}{2}\left[f(x) + f(-x)\right] + \frac{1}{2}\left[f(x) - f(-x)\right]$$

Let $\frac{1}{2}\left[f(x) + f(-x)\right] = g(x)$

& $\frac{1}{2}\left[f(x) - f(-x)\right] = h(x)$

So that $f(x) = g(x) + h(x)$

Now $g(-x) = \frac{1}{2}[f(-x) + f(x)] = g(x)$

i.e $g(x)$ is even

& $h(-x) = \frac{1}{2}[f(-x) - f(x)] = -h(x)$

i.e $h(x)$ is odd.

Example - 3 :—

- Find whether the following functions are even or odd or neither of them.

(i) $x\sin x + \cos x + x^2\cosh x$

(ii) $\log\left[\frac{1-x}{1+x}\right]$

(iii) $x\cosh x + x^3\sinh x$.

Solution :—

(i) Let $f(x) = x\sin x + \cos x + x^2\cosh x$

$$f(-x) = (-x)\sin(-x) + \cos(-x) + (-x)^2\cosh(-x) \quad \left[\begin{array}{l}\sin(-x) = -\sin x \\ \cos(-x) = \cos x \\ \cosh(-x) = \cosh x\end{array}\right.$$

$$= x\sin x + \cos x + x^2\cosh x$$

$$= f(x)$$

(ii) Let $f(x) = \log\left[\frac{1-x}{1+x}\right]$

$$f(x) = \log(1-x) - \log(1+x)$$

$$f(-x) = \log(1+x) - \log(1-x)$$

$$= -[\log(1-x) + \log(1+x)]$$

$\Rightarrow f(-x) = -f(x)$

$\therefore f(x)$ is odd.

(iii) Let $f(x) = x\cosh x + x^3 \sinh x$

$$f(-x) = -x\cosh(-x) + (-x)^3 \sinh(-x) \quad \left[\because \begin{array}{l} \cosh(-x) = \cosh x \\ \sinh(-x) = -\sinh x \end{array}\right.$$

$$= -x\cosh x + x^3 \sinh x$$

$\therefore f(x)$ is neither even or odd.

$\boxed{\text{Class-2}}$

Introduction To Fourier Series :—

- Fourier Series is an infinite series representation of periodic function in terms of sines & cosines.
- We know that Taylor's series expansion is valid only for functions which are continuous & differentiable.
- But fourier series is possible for continuous function, periodic functions & functions discontinuous in their value & derivatives.
- Fourier series is useful to solve ordinary & partial differential equations, particularly with periodic functions appearing as non-homogeneous terms.

- Suppose that a given function $f(x)$ defined in $[-\pi, \pi]$ or $[0, 2\pi]$ or in any other interval can be expressed as a trigonometric series as

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + a_n \cos nx + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots + b_n \sin nx + \cdots$$

$$= \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$$
$$+ \cdots + (a_n \cos nx + b_n \sin nx) + \cdots$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where a's & b's are constant within a desired range of values of the variable.

- Such a series is known as fourier series for $f(x)$ & the constants a_0, a_n, b_n $(n = 1, 2, 3 \cdots)$ are called Fourier Coefficients of $f(x)$.

- Every term of (a_n) except the first, has period 2π & hence any function represented by a series ~~is known as~~ of the above form will also be periodic with period 2π.

CLASS-3

Dirichlet Conditions :-

- Suppose that

  $f(x)$ is defined & single valued except possibly at a finite number of points in $(-\pi, \pi)$

  $f(x)$ is periodic with period 2π

  $f(x)$ & $f'(x)$ are piecewise continuous in $(-\pi, \pi)$

- Then the above series converges to

  $f(x)$, if x is a point of continuity.

  $\dfrac{f(x+0)+f(x-0)}{2}$, if x is a point of discontinuity.

- Therefore the value of $f(x)$ at any point of continuity x in $[-\pi, \pi]$ is given by

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

- The value of $f(x)$ at any point of discontinuity x in $(-\pi, \pi)$ is given by

$$\frac{f(x+0)+f(x-0)}{2}$$

- The above condition imposed on $f(x)$ are sufficient but not necessary i.e if the conditions are satisfied the convergence is guaranteed.

- However, if they are not satisfied the series may or may not converge.

The conditions above are generally satisfied in cases which arise in ~~sci~~ science or engineering.

Let the function $f(x) = x^2$ be defined in the interval $-\pi < x < \pi$

Let its fourier series be

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

At $x = 0$ ('0' lies within $-\pi < x < \pi$ & hence it is a point of ~~dis~~continuity) the sum of the fourier series (i) equal to $f(0)$

$$\therefore f(0) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow 0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Suppose $f(x)$ is defined in the interval $-\pi < x < \pi$, then at the points of discontinuities (at $x = \pi$ or $x = -\pi$), the sum of the fourier series is equal to the arithematic mean of the value of $f(x)$ at $x = \pi$ & $x = -\pi$.

Sum of the series at $x = \pi$ is equal to $\frac{f(\pi) + f(-\pi)}{2}$

Suppose the function $f(x)$ is defined by

$$f(x) = \begin{cases} -\pi, & \text{in } -\pi < x < 0 \\ x, & \text{in } 0 < x < \pi \end{cases}$$

Then at the point of discontinuity $x=0$ (which is in the middle of the given interval) the sum of the fourier series converges (or equals) to the average value of the right hand limit & left hand limit of the given function at $x=0$.

sum of the series at $x=0$ is equal to $\frac{f(0+0)+f(0-0)}{2}$

Here sum of the series $= \frac{0-\pi}{2} = \frac{-\pi}{2}$

If x is a point of continuity, then the sum of the fourier series is equal to $f(x)$

i.e Sum of the series $= f(x)$

$\boxed{\text{CLASS-4}}$: —

- Some Important formulae : —

- $\int_{c}^{c+2\pi} \cos nx \, dx = 0 \; (n \neq 0)$
- $\int_{c}^{c+2\pi} \sin nx \, dx = 0 \; (n \neq 0)$
- $\int_{c}^{c+2\pi} \cos mx \cos nx \cdot dx = 0 \; (m \neq n)$
- $\int_{c}^{c+2\pi} \cos^2 nx \cdot dx = \pi \; (n \neq 0)$
- $\int_{c}^{c+2\pi} \sin mx \cos nx \cdot dx = 0 \; (m \neq n)$
- $\int_{c}^{c+2\pi} \sin nx \cdot \cos nx \cdot dx = 0$

- $$\int_{c}^{c+2\pi} \sin mx \sin nx \cdot dx = 0 \; (m \neq n)$$

- $$\int_{c}^{c+2\pi} \sin^2 nx \cdot dx = \pi \; (n \neq 0)$$

- **CLASS-15**

- **Determination of Fourier Coefficient :—**

- **Euler's Formula :—**

The Fourier series for the function $f(x)$ in the interval $c < x \leq c+2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

where
$$a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \cdot dx$$

$$a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \cdot \cos nx \cdot dx$$

$$b_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \sin nx \, dx .$$

These values a_0, a_n, b_n are known as Euler's formula

Proof:— Let $f(x)$ be represented in the interval $[c, c+2\pi]$ by the fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Let us assume that the series is uniformly convergent in the interval $c \leq x \leq c+2\pi$. Then the series can be integrated term by term.

To evaluate a_0 :—

Integrating both sides of eq^n (1) from $x = c$ to $x = c+2\pi$

we get,

$$\int_c^{c+2\pi} f(x)\cdot dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \sum_{n=1}^{\infty}\left[a_n \int_c^{c+2\pi} \cos nx \cdot dx + b_n \int_c^{c+2\pi} \sin nx\right.$$

$$= \frac{a_0}{2}(c+2\pi - c) + \sum_{n=1}^{\infty}\left[a_n\left(\frac{\sin nx}{n}\right)_c^{c+2\pi} + b_n\left(-\frac{\cos nx}{n}\right)_c^{c+2}\right.$$

$$= a_0\pi + \sum_{n=1}^{\infty}[a_n\cdot 0 - b_n\cdot 0]$$

$$= a_0\pi$$

$$\Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cdot dx}$$

To Evaluate a_n :—

Now multiply both sides of equation (1) by $\cos mx$ integrating from $x = c$ to $c+2\pi$, we get,

$$\int_c^{c+2\pi} f(x)\cos mx\cdot dx = \frac{a_0}{2}\int_c^{c+2\pi}\cos mx\cdot dx + \sum_{n=1}^{\infty} a_n\left[\int_c^{c+2\pi}\cos nx\cdot \cos mx\cdot dx\right]$$

$$+ \sum_{n=1}^{\infty} b_n\left[\int_c^{c+2\pi}\sin nx\cos mx\cdot dx\right]$$

The first & 3rd integrals on the R.H.L are always but the second integral equal to π when $m = n$ otherwise vanishes when $m \neq n$, hence

$$\Rightarrow \int_c^{c+2\pi} f(x)\cos nx \cdot dx = \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n\int_c^{c+2\pi} 2\cos^2 nx \cdot dx\right]$$

$$= \frac{a_n}{2}\int_c^{c+2\pi}(1+\cos 2nx)\,dx$$

$$= \frac{a_n}{2}\left(x+\frac{\sin 2nx}{2n}\right)_c^{c+2\pi}$$

$$= \frac{a_n}{\cancel{2}} \times \cancel{2}\pi$$

$$= a_n\pi$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx \cdot dx$$

<u>To Evaluate b_n:-</u>

Multiplying the eq^n ① with $\sin mx$ & Integrating $x = c$ to $x = c+2\pi$, we get

$$\int_c^{c+2\pi} f(x)\sin mx \cdot dx = \frac{a_0}{2}\int_c^{c+2\pi}\sin mx \cdot dx + \sum_{n=1}^{\infty} a_n \int_c \cos nx \cos \sin mx\, dx$$

$$+ b_n \int_c^{c+2\pi} \sin nx \sin mx \cdot dx$$

The first two integrals on the right hand side equals to zero, but the third equals to π when $m=n$ otherwise vanishes.

$$\int_c^{c+2\pi} f(x)\sin nx \cdot dx = \frac{a_0}{2}(0) + a_n(0) + \frac{1}{2}b_n\int_c^{c+2\pi} 2\sin^2 nx \cdot dx$$

$$= \frac{1}{2}b_n\int_c^{c+2\pi}(1-\cos 2nx)\,dx$$

$$= \frac{1}{2}b_n\left(x-\frac{\sin 2nx}{2\pi}\right)_c^{c+2n\pi} = \pi b_n$$

~~3-D Geometry~~

$$\Rightarrow b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx \, dx \quad \text{——— (4)}$$

eq^n (2), (3), (4) are called Euler's formula.

Note:-

(1) If $f(x)$ is to be expanded as a fourier series in the interval $0 \le x \le 2\pi$, put $c=0$, then the formula reduces to,

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\,.dx$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\,.dx$$

$$b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx\,.dx$$

(2) If $f(x)$ is to be expanded as a fourier series in $[-\pi,\pi]$, put $c=-\pi$, the interval becomes $-\pi \le x$ & the formula reduces to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,.dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,.dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,.dx$$

$\boxed{\text{CLASS - 6}}$

Problems

Ex-1: Find the fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$

Ans: Let $x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

We know that,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x).dx \ , \ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx . dx \ , \ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx . dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 . dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{8\pi^3}{3} - 0\right] \Rightarrow \boxed{a_0 = \frac{8}{3}\pi^2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx . dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx . dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$\Rightarrow \boxed{a_n = \frac{4}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx . dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx \, dx$$

$$\Rightarrow \boxed{b_n = \frac{-4\pi}{n}}$$

Substituting the value of a_0, a_n, b_n in (1)

$$x^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty} \frac{4\pi}{n}\sin nx$$

Ex-2:- Find the fourier's series to represent the function $f(x) = e^x$, for $-\pi < x < \pi$ & hence derive a series for s

Ans:- $$e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x.dx = \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]$$

$$\Rightarrow \boxed{a_0 = \frac{2}{\pi}\sinh \pi}$$

& $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx.dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$\boxed{a_n = \frac{2(-1)^n \sinh \pi}{\pi(1+n^2)}}$$

Again $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx.dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x}\sin nx \cdot dx$$

$$= \frac{1}{\pi}\left[\frac{e^{x}}{1+n^2}(\sin nx - n\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{x}(\sin n\pi - n\cos n\pi) - e^{-\pi}(\sin(-n\pi) - n\cos(-n\pi)\right]$$

$$= \frac{2(-1)^{n+1}}{\pi(1+n)}(e^{\pi} - e^{-\pi})$$

$$\Rightarrow b_n = \frac{2n(-1)^{n+1}\sinh\pi}{\pi(1+n^2)}$$

Substituting all the above values in (1) we get

$$\boxed{e^{x} = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos nx + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{1+n^2}n\sin nx\right]}$$

Practice Set - 1

(1) Obtain the fourier series expansion of $f(x) = kx(\pi - x)$ given that in $0<x<2\pi$, where k is constant.

(2) Obtain the fourier series expansion of $f(x)$ given that $f(x) = (\pi - x)^2$ in $0<x<2\pi$ & deduce the value of

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

(3) Find the fourier series of period 2π for the function $f(x) = x^2 - x$ in $(-\pi, \pi)$. Hence deduce the sum of the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{12}$

CLASS-7

Functions Having points of Discontinuity :—

In deriving the Euler's formula for a_0, a_n & b_n, it was that $f(x)$ is continuous, Instead a function may have a finite no of discontinuities, Even then such a func is expressible as a fourier series. for instance,

Let the function $f(x)$ be defined by,

$$f(x) = \phi(x) \,,\quad c < x < x_0$$

$$= \psi(x) \,,\quad x_0 < x < c+2\pi$$

where x_0 is the point of discontinuity in $[c, c+2\pi$

In such case, we obtain the fourier series in usua

where
$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0}\phi(x)\cdot dx + \int_{x_0}^{c+2\pi}\psi(x)\cdot dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0}\phi(x)\cos nx\cdot dx + \int_{x_0}^{c+2\pi}\psi(x)\cos nx\cdot dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0}\phi(x)\sin nx\cdot dx + \int_{x_0}^{c+2\pi}\psi(x)\sin nx\cdot dx\right]$$

- Here at a point of finite discontinuity $x = x_0$, there finite jump in the value of function $f(x)$ at $x = x_0$.

- In $[c, c+2\pi]$, then the right hand side converges $f(x)$ if x is a point of continuity of $f(x)$ & converg to $\frac{1}{2}[f(x+0) + f(x-0)]$ if x is a point of discontinuity of $f(x)$. This result is useful to determine the su of certain infinite series.

Example:- Expand $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$ as a fourier series.

The fourier series of $f(x)$ in the interval $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x).dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x).dx + \int_{\pi}^{2\pi} f(x).dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} dx + \int_{\pi}^{2\pi} 0.dx\right]$$

$$= \frac{1}{\pi} \times \pi = 1$$

$\Rightarrow a_0 = 1$.

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} 1.\cos nx.dx + \int_{\pi}^{2\pi} 0.\cos nx.dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\sin nx}{n}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{-\sin n\pi}{n} + \frac{\sin 0}{n}\right]$$

$$= \frac{1}{n\pi} \times 0$$

$\Rightarrow a_n = 0$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} 1.\sin nx.dx + \int_{\pi}^{2\pi} 0.\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{-1}{n\pi}\left[(-1)^n - 1\right].$$

$\therefore b_n = \frac{-1}{n\pi}\left[(-1)^n - 1\right]$

CLASS-8

Problems :-

1) Find the Fourier series of the function $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, Hence evaluate $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.}$

solution: Given that,

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$$

Fourier series is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cdot dx + \int_{0}^{\pi} f(x) \cdot dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x \cdot dx = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{-1}{\pi}\left[\cos \pi\right.$$

$$= \frac{-1}{\pi}[-1-1] = \boxed{\frac{2}{\pi} = a_0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \cdot dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \underbrace{f(x) \cos nx \cdot dx}_{0} + \int_0^{\pi} f(x) \cos nx \cdot dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx \cdot dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\sin (n+1)x - \sin (n-1)x\right] dx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi}[\sin (n+1)x]\,dx - \int_0^{\pi}(\sin (n-1)x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{-\cos (n+1)x}{n+1} + \frac{\cos (n-1)}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{(-1)(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[(-1)^n\left(\frac{-2}{n^2-1}\right) + \left(\frac{-2}{n^2-1}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{-2}{n^2-1}\left((-1)^n+1\right)\right]$$

$$= \frac{1}{\pi(n^2-1)}\left(1+(-1)^n\right) \quad n \neq 1$$

$\therefore a_n = 0$ if n is odd.

$= \frac{-2}{\pi(n^2-1)}$ if n is even

for $n=1$, $a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cdot \cos x \cdot dx$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin 2x \cdot dx = \frac{1}{2\pi}\left(\frac{-\cos 2x}{2}\right)_0^{\pi}$$

$$= 0$$

similarly, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \cdot dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x) \sin nx \, dx + \int_0^{\pi} f(x) \sin nx \cdot dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_0^{\pi} f(x) \sin nx \cdot dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin nx \cdot \sin x \cdot dx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} \cos(n-1)x \cdot dx - \int_0^{\pi} \cos(n+1)x \cdot dx\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{\sin(n-1)x}{n-1}\right]_0^{\pi} - \left(\frac{\sin(n+1)x}{n+1}\right)_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left(\frac{\sin[(n-1)\pi]}{n-1} - \frac{\sin(n+1)\pi}{n+1} - 0 + 0\right)$$

$\therefore b_n = 0 \ \forall \ n \neq 1$

(21)

for $n=1$, we have $b_1 = \frac{1}{\pi}\int_0^{\pi}(1-\cos 2x)\,dx$

$$= \frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$\Rightarrow \boxed{b_1 = \frac{1}{2}}$$

Substituting all the values in eqⁿ ①, we get

$$f(x) = \frac{1}{\pi} - \sum_{n=2,4,6} \frac{2}{\pi(n^2-1)}\cos nx + \frac{1}{2}\sin x$$

Thus n takes only even values

let $n = 2m$

where $m = 1, 2, 3 \ldots$

Now eqⁿ ④ becomes

$$f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \sum_{m=1}^{\infty} \frac{2}{\pi(4m^2-1)}\cos 2mx.$$

When $x=0$, the fourier series in eqⁿ ⑤ is convergent

$f(0) = 0$

Thus when $x=0$, the fourier series in eq ⑤ becom

$$\Rightarrow 0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{4m^2-1}$$

$$\Rightarrow \frac{1}{\pi} = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)(2m+1)}$$

$$\Rightarrow \frac{1}{\pi} = \frac{2}{\pi}\left[\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right]$$

$$\Rightarrow \boxed{\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots}$$

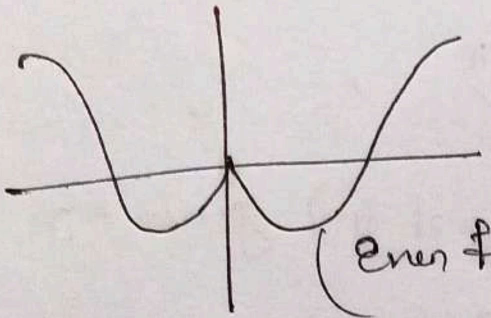
Ans

CLASS - 9

Even Function :—

- If $f(-x) = f(x)$, then $f(x)$ is called Even function e.g. $\cos x$, x^2, x^4+3x^2 are all even functions.

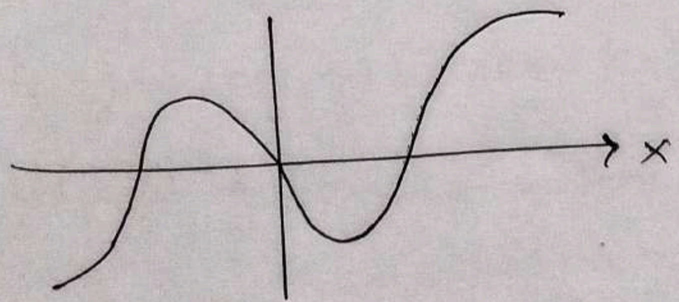
- The graph of even function is symmetrical about Y-axis (i.e it is a mirror image)



(Even fun^n)

Odd function :—

- If $f(-x) = -f(x)$, then $f(x)$ is said to be odd function
- E.g: x^3, $\sin x$.
- The graph of an odd function is symmetrical about the origin



(odd function)

Properties :—

① The product of two even function is even
② The product of even & odd function is Odd

(22) Practice Set - 2 :-

(1) Obtain Fourier series for

$$f(x) = \begin{cases} x & \text{in } -\pi < x < 0 \\ 0 & \text{in } 0 < x < \frac{\pi}{2} \\ x - \frac{\pi}{2} & \text{in } \frac{\pi}{2} < x < \pi \end{cases}$$

(2) Find the fourier series of the following function

$$f(x) = \begin{cases} -\cos x, & -\pi < x < 0 \\ \cos x & 0 < x < \pi \end{cases}$$

(3) Find the fourier series of $f(x)$ given by

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases}$$

- The sum of two even function is even.
- The sum of two odd function is odd.
- $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{when } f(x) \text{ is even function} \\ 0, & \text{when } f(x) \text{ is odd function} \end{cases}$$

Examples :—

① $f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$

$\Rightarrow f(x) = x^2$ is an even function

② $f(x) = x^3$.

$f(-x) = (-x)^3 = -x^3 = -f(x)$

$\therefore x^3$ is an odd function

③ $f(x) = \cos x$

$\Rightarrow f(-x) = \cos(-x) = \cos x = f(x)$

$\Rightarrow \cos x$ is an even function

④ $f(x) = \sin x$.

$f(-x) = \sin(-x) = -\sin x = -f(x)$.

$\therefore \sin x$ is an odd function.

CLASS-10

- Fourier Series for Even & Odd function :-

We know that a function $f(x)$ defined in $(-\pi, \pi)$ can be represented by the fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$

Case-I :- When $f(x)$ is even function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).dx$$

$$\boxed{a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x).dx}$$

and $f(x) = \cos nx$ is an even function

Hence $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$

$$\boxed{a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx.dx}$$

Again $\sin nx$ is an odd function

$\Rightarrow$ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 0$

Thus, if $f(x)$ is defined in $[-\pi, \pi]$ & $f(x)$ is an even function, $f(x)$ can be expanded as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \cdot dx$, $n = 0, 1, 2 \dots$

& $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) \cdot dx$

Problem:-

- Expand the function $f(x) = x^2$ as a fourier series in $[-\pi, \pi]$. prove that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$, hence deduce that,

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$.

Soln:- $f(x) = x^2$

$f(-x) = (-x)^2$

$f(-x) = x^2 = f(x)$

$\therefore f(x)$ is even function in $[-\pi, \pi]$

Hence the fourier series expansion, the sine term absent

$$\therefore x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{——— (1)}$$

(26)

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 \cdot dx$

$$= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{\pi^3}{3} - 0\right)$$

$$\boxed{a_0 = \frac{2\pi^2}{3}} \quad \text{--- (2)}$$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \cdot dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \cdot dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2(0) + \frac{2\pi(-1)^n}{n^2} + 0\right]$$

$$\Rightarrow \boxed{a_n = \frac{4}{n^2}(-1)^n} \quad \text{--- (3)}$$

Now putting the value of a_0, & a_n in (1)

$$\Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$= \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos nx$$

$$\boxed{x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right)} \text{ --- }$$

(1) Now putting $x = 0$ in (4)

$$0 = \frac{\pi^2}{3} - 4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$$

$$\Rightarrow \boxed{1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}} \quad \text{--- (6)}$$

(2) Putting $x = \pi$ in (4) we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left(\cos\pi - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \frac{\cos 4\pi}{4^2} + \cdots\right)$$

$$\Rightarrow \boxed{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}} \quad \text{--- (7)}$$

Now adding ⑥ & ⑦

$$\left(1-\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)+\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)=\frac{\pi^2}{12}+\frac{\pi^2}{6}$$

$$\Rightarrow\left(2+\frac{2}{3^2}+\frac{2}{5^2}+\cdots\right)=\frac{2\pi^2+\pi^2}{12}$$

Now dividing 2 on both sides

$$\Rightarrow \boxed{\left(1+\frac{1}{3^2}+\frac{1}{5^2}\cdots\right)=\frac{3\pi^2}{24}=\frac{\pi^2}{8}}$$

Ans

CLASS-11

Case-2 :- When $f(x)$ is an odd function in $(-\pi,\pi)$

Here $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot dx=0$

Again $\cos nx$ is even, but $f(x)$ is odd.

$\Rightarrow f(x)\cos nx$ is odd

$\therefore\ a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx=0$

Again $\sin nx$ is odd & $f(x)$ is odd function

$\Rightarrow f(x)\sin nx$ is even function

$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$=\frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\cdot dx$

Thus, if a function $f(x)$ defined in $(-\pi, \pi)$ is odd, wh
its fourier expansion contains only sine terms.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \cdot dx$

Problem:

Q/ Find the fourier expansion of $f(x) = x\cos x$, $0 < x$

Soln The given function is

$$f(x) = x\cos x \text{ as } 0 < x < 2\pi$$

But $f(x)$ is an odd function of x.

Thus the fourier series is sine series.

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin x.$$

Where $b_n = \frac{1}{\pi}\int_0^{2\pi} x\cos x \sin nx \, dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2x\cos x \cdot \sin nx \cdot dx$$

$$= \frac{1}{\pi} \times \frac{1}{2}\int_0^{2\pi} x\left[\sin(n+1)x + \sin(n-1)x\right] dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\sin(n+1)\,dx + \int_0^{2\pi} x\sin(n-1)x \cdot dx\right]$$

$$= \frac{1}{2\pi}\left[-x\frac{\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} - x\frac{\cos(n-1)x}{n-1} + \frac{\sin}{(n}\right.$$

$$= \frac{1}{2\pi}\left[-2\pi\left(\frac{\cos(n+1)2\pi}{n+1} + 0\right)\right] - \frac{1}{2\pi}\left[\frac{\cos(n-1)2\pi}{n-1} + 0\right]$$

$$= \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} = (-1)^n\left[\frac{(n+1)+(n-1)}{(n+1)(n-1)}\right] = \frac{2n(-1)}{(n-1)(n+}$$

When $n=1$, b_n becomes infinite. So, finding b_1,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \cdot dx$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos x \cdot \sin x \cdot dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2x\cos x \cdot \sin x \cdot dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x \cdot dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) + \frac{\sin 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\sin 4\pi}{4} - \frac{2\pi\cos 4\pi}{2}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi}{2}\right]$$

$$\therefore \boxed{b_1 = \frac{-1}{2}}$$

Substituting b_1 & b_n in eq^n 1, we get

$$\therefore f(x) = \frac{-1}{2}\sin x + \sum_{n=2}^{\infty} \frac{2n(-1)^n}{(n-1)(n+1)} \sin nx$$

Problem-2: Obtain Fourier series for the function $f(x)$ given by

$$f(x) = \begin{cases} \frac{-1}{2}(\pi+x), & \text{for } -\pi < x \le 0 \\ \frac{1}{2}(\pi+x), & \text{for } 0 \le x < \pi \end{cases}$$

Solution: $f(x) = \begin{cases} \frac{-1}{2}(\pi+x), & \text{for } -\pi < x \le 0 \\ \frac{1}{2}(\pi+x) & \text{for, } 0 \le x < \pi \end{cases}$

$f(x) = \frac{-1}{2}(\pi + x)$ in $(-\pi, 0)$

since , $f(-x) = \frac{-1}{2}(\pi \mp x)$ in $(0, \pi)$

$= -f(x)$ in $(0, \pi)$

$f(x) = \frac{1}{2}(\pi - x)$ in $(0, \pi)$

$f(-x) = \frac{1}{2}(\pi + x)$ in $(0, \pi)$

$= -f(x)$ in $(-\pi, 0)$

$\therefore f(x)$ is an odd function in $(-\pi, \pi)$

Hence $a_0 = a_n = 0$, now let .

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Hence $a_0 = a_n = 0$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}(\pi - x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{-1}{\pi}\left[(\pi - x)\left(\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$b_n = \frac{-1}{\pi}\left[0 + 0 - \pi\left(\frac{\cos 0}{n}\right) - 0\right]$$

$$b_n = \frac{-1}{\pi}\left(\frac{-\pi}{n}\right)$$

$$b_n = \frac{1}{n}.$$

Now substituting b_n in eqn ①, we get

$$\boxed{f(x) = \sum_{n=1} \frac{1}{n} \sin nx}$$

$\boxed{\text{CLASS - 12}}$

• Problem Practice

(Q1) Find the Fourier series to represent the function $f(x) = \sin x$, $-\pi < x < \pi$

Ans: Since $f(x) = |\sin x|$ is an even function

$\Rightarrow$ Its Fourier series consists of cosine terms only.

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x) \cdot dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin x \cdot dx$$

$$= \frac{2}{\pi}\Big[-\cos x\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\Big[-(-1)+1\Big]$$

$$\boxed{a_0 = \frac{4}{\pi}} \quad \&$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \cdot dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx \cdot dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin x \cdot \cos nx \cdot dx$$

(b) Since $f(x)$ is an even function in $(-\pi, \pi)$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{———} \quad (1)$$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$$

$$\cancel{a_0 = 0}$$

$$a_0 = \frac{2}{\pi}\left[x - \frac{2x^2}{2\pi}\right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi}\left[\pi - \cancel{\frac{2\pi^2}{2\pi}} - 0\right]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) + \frac{2}{\pi}\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 - \frac{2}{\pi}\frac{(-1)^n}{n^2} - 0 + \frac{2}{\pi}\frac{1}{n^2}\right]$$

$$= \frac{4}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$a_n = \frac{8}{n^2\pi^2}$ if n is odd.

$a_n = 0$ if n is even

$\therefore a_1 = \frac{8}{\pi^2}, a_3 = \frac{8}{3^2\pi^2}, a_5 = \frac{8}{5^2\pi^2} \cdots$ & $a_2 = a_4 = a_6 = \cdots = 0$

Substituting the values in ①

$$\boxed{f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]} \quad \text{—— ②}$$

Putting $x=0$, in ②

$$f(0) = 1 = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow \boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}}$$

Practice Questions:-

① Obtain the fourier series of $f(x) = x + x^2$ in $(-\pi, \pi)$

② Obtain the fourier series of $f(x) = x\sin x$ in $(0, 2\pi)$

③ If $f(x) = x$ in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ & $f(x) = 0$ in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. find the fourier series of $f(x)$, Deduce that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

④ Obtain the fourier series expansion of $f(x) = e^{ax}$ in $(0, 2\pi)$

⑤ Obtain the fourier series of $f(x) = \pi - x/2$ in $(0, 2\pi)$

Deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Assignment

1. Find the fourier series of $f(x) = x$ in the interval $(0, 2\pi)$

2. Find the fourier series of $f(x) = e^{-x}$ in the interval $(0, 2\pi)$

3. Find the fourier series of $f(x) = x \sin x$ in the interval $(-\pi, \pi)$

4. Find the fourier series of $f(x) = e^{ax}$ in the interval $(-\pi, \pi)$

5. Find the fourier series of $f(x) = x^2$ in $(0, 2\pi)$ & hence deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

6. Find the fourier series of $f(x) = |x|$ in the interval $[-\pi, \pi]$. Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

7. Answer the following Questions.

(a) Explain periodic function with examples.

(b) State Dirichlet's conditions for a function to be expanded as a fourier series.

(c) State whether $y = \tan x$ can be expressed as a Fourier series. if so how? if not why?

(d) Write the formulae for fourier constants for $f(x)$ in the interval $(-p, p)$

(e) To what value does the sum of Fourier series of $f(x)$ converge at the point of discontinuity at $x = a$

(f) find the constant term a_0 in the fourier series corres-ponding to $f(x) = x - x^3$ in $(-\pi, \pi)$

(g) State the convergence condition on Fourier Series.