Vectors

Introduction :-

In our real life situation we deal with physical quantities such as distance, speed, temperature, volume etc. These quantities are sufficient to describe change of position, rate of change of position, body temperature or temperature of a certain place and space occupied in a confined portion respectively.

We have also come across physical quantities such as displacement, velocity, acceleration, momentum etc. which are of different type in comparison to above. Consider the figure-1, where A, B, C are at a distance 4Km from P. If we start from P, then covering 4Km distance is not sufficient to describe the destination where we reach after the travel, So here the end point plays an important role giving rise the need of direction. So we need to study about direction of a quantity, along with magnitude.

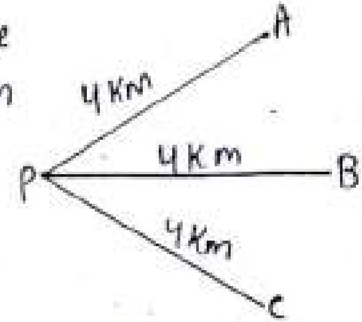


Fig - 1

Objective

After completion of the topic you are able to :-

(i) Define and distinguish between scalars and vectors.

(ii) Represent a vector as directed line segment.

(iii) classify vectors in to different types.

(iv) Resolve vector along two or three manually perpendicular axes.

(v) Define dot product of two vectors and explain its geometrical meaning.

(vi) Define cross product of two vectors and apply it to find area of triangle and parallelogram.

Scalars and Vectors

All the physical quantities can be divided into two types
(i) Scalar quantity or Scalar
(ii) Vector quantity or Vector

Scalar quantity :- The physical quantities which require only magnitude for its complete specification is called as Scalar quantities.
Examples :- Speed, mass, distance, velocity, volume etc.

Vector :- A directed line segment is called as vector.

Vector quantities :- A physical quantity which requires both magnitude & direction for its complete specification and satifies the law of vector addition is called as vector quantities.
Examples :- Displacement, Force, acceleration, velocity, momentum etc.

Representation of Vector :- A vector is directed line segment $\overrightarrow{AB}$ where A is the initial point and B is the terminal point and direction is from A to B. (see fig-2)

Similarly $\overrightarrow{BA}$ is a directed line which represents a vector having initial point B and Terminal A.

Fig-2

Fig-3

Notation :- A vector quantity is always represented by an arrow ($\rightarrow$) mark over it or by bar (-) over it. For example $\overrightarrow{AB}$. It is also represented by a single small letter with an arrow or bar mark over it. For example $\vec{a}$.

Magnitude of $\overrightarrow{AB} = |\overrightarrow{AB}| =$ Length $AB = AB$.

Types of Vector :- Vectors are following types

(1) Null vector or Zero Vector or void vector – A vector having zero magnitude and arbitrary direction is called as a null vector and is denoted by $\vec{0}$.

Clearly, a null vector has no definite direction. If $\vec{a} = \vec{AB}$, then $\vec{a}$ is a null (or zero) vector if $|\vec{a}| = 0$ i.e. if $|\vec{AB}| = 0$

(2) Proper vector – Any non zero vector is called as a proper vector. If $|\vec{a}| \neq 0$ then $\vec{a}$ is a proper vector.

(3) Unit Vector – A vector whose magnitude is unity is called a unit vector. Unit vectors are denoted by a small letter ^ over it. For example $\hat{a}$. $|\hat{a}| = 1$.

Note :- The unit vector along the direction of a vector $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

(4) Co-initial vectors :- Vectors having the same initial point are called co-initial vector.

In figure-4, $\vec{OA}$, $\vec{OB}$, $\vec{OC}$, $\vec{OD}$ and $\vec{OE}$ are co-initial vectors.

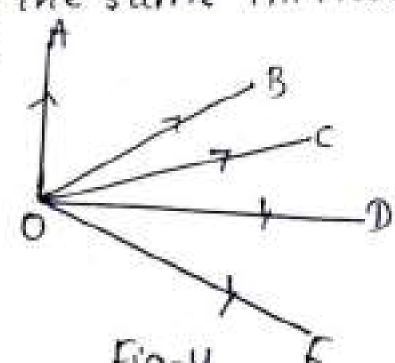


Fig-4

(5) Like and Unlike Vectors :- Vectors are said to be like if they have same direction and unlike if they have opposite direction.

6) <u>Co-Linear Vectors</u> :- Vectors are said to be colinear or parallel if they have the same line of action. In figure-5 $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are co-linear.

Fig 5

7) <u>Parallel Vectors</u> :- Vectors are said to be parallel if they have same line of action or have line of action parallel to one another. In Fig-6 the vectors are parallel to each other.

Fig 6

8) <u>Co-planner vectors</u> :- Vectors are said to be co-planner if they lies on the same plane. In fig-7 vector $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanner.

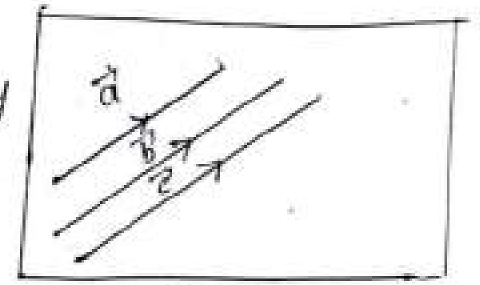


Fig.7

9) <u>Negative of a Vector</u> :- A vector having same magnitude but opposite in direction to that of a given vector is called negative vector of that vector. If $\vec{a}$ is any vector then negative vector of it is written as $-\vec{a}$ and $|\vec{a}| = |-\vec{a}|$ but both have direction opposite to each other as shown in Fig-8.

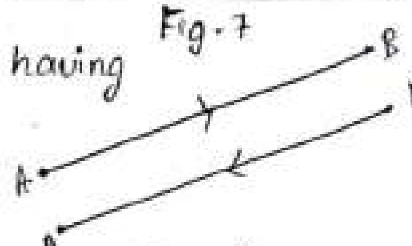
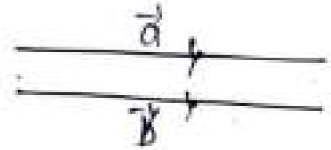


Fig-8.

10) Equal Vectors :- Two Vectors are said to be equal if they have same magnitude as well as same direction.



Thus $\vec{a} = \vec{b}$.

Notes :- Two vectors can not be equal

(i) If they have different magnitude

(ii) If they have inclined supports

(iii) If they have different sense.

Vector operations

Addition of vectors :-

Triangle law of vector addition:- The law states that if two vectors are represented by the two sides of a triangle taken in same order their sum or resultant is represented by the 3rd side of the triangle with direction in reverse order.

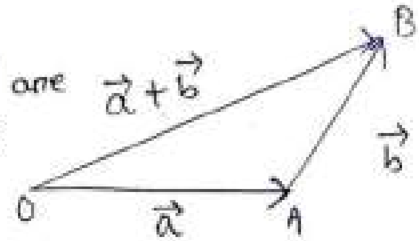


Fig-10

As shown in Figure-10 $\vec{a}$ and $\vec{b}$ are two vectors represented by two sides OA and AB of a triangle ABC in same order. Then the sum $\vec{a}+\vec{b}$ is represented by the third side OB taken in reverse order. i.e the vector $\vec{a}$ is represented by the directed segment $\overrightarrow{OA}$ and the vector $\vec{b}$ be the directed segment $\overrightarrow{AB}$, so that the terminal point A of $\vec{a}$ is the initial point of $\vec{b}$. Then $\overrightarrow{OB}$ represents the sum (or resultant) $(\vec{a}+\vec{b})$. Thus $\overrightarrow{OB} = \vec{a}+\vec{b}$

Note-1- The method of drawing of triangle in order to define the vector sum $(\vec{a}+\vec{b})$ is called triangle law of addition of the vectors.

Note-2- Since any side of a triangle is less than the sum of the other two sides. $|\overrightarrow{OB}| \neq |\overrightarrow{OA}| + |\overrightarrow{AB}|$

Parallelogram law of vector addition-

If $\vec{a}$ and $\vec{b}$ are two vectors represented by two adjacent side of a parallelogram in a magnitude and direction, then their sum (resultant) is represented in magnitude and direction by the diagonal which is passing through the common initial point of the two vectors. As shown in fig-11 if OA is $\vec{a}$ and AB is $\vec{b}$ then OB diagonal represent $\vec{a}+\vec{b}$.

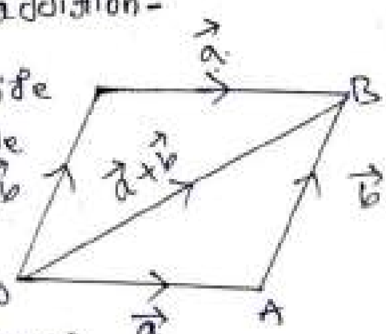


Fig - 11

i.e. $\vec{a} + \vec{b} = \vec{OA} + \vec{AB}$

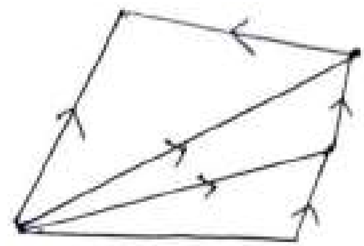


Fig 12

Polygon law of vector addition:- If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are the four sides of a polygon in the same order then their sum is represented by the last side of the polygon taken in opposite order as shown in Fig-12.

Subtraction of two Vectors

If $\vec{a}$ and $\vec{b}$ are two given vectors then the subtraction of $\vec{b}$ from $\vec{a}$ denoted by $\vec{a} - \vec{b}$ is defined as addition of $-\vec{b}$ with $\vec{a}$. i.e., $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

Properties of vector addition:-

(i) Vector addition is commutative i.e., if $\vec{a}$ & $\vec{b}$ are any two vectors then $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(ii) Vecotor addition is associative i.e., if $\vec{a}, \vec{b}, \vec{c}$ are any three vectors then $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

(iii) Existence of additive identity i.e., for any vector $\vec{a}$, $\vec{0}$ is the additive identity i.e., $\vec{0} + \vec{a} = \vec{a} + \vec{0} = \vec{a}$ where $\vec{0}$ is a null vector.

(iv) Existence of additive inverse:- If $\vec{a}$ is any non-zero vector then $-\vec{a}$ is the additive inverse of $\vec{a}$ so that $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$

Multiplication of a vector by a scalar

If $\vec{a}$ is a vector and K is a non-zero scalar then the multiplication of the vector $\vec{a}$ by the scalar K is a vector denoted by $K\vec{a}$ or $\vec{a}K$ whose magnitude |K| times that of $\vec{a}$.

i.e $K\vec{a} = |K| \times |\vec{a}|$

$= K \times |\vec{a}|$ if $K \geq 0$

$= (-K) \times |\vec{a}|$ if $K < 0$

The direction of $K\vec{a}$ is same as that of $\vec{a}$ if K is positive and opposite as that of $\vec{a}$ if K is negative. $K\vec{a}$ and $\vec{a}$ are always parallel to each other.

Properties of scalar multiplication of Vectors:-

If h and K are scalars and $\vec{a}$ and $\vec{b}$ are given vectors then

(i) $K(\vec{a}+\vec{b}) = K\vec{a} + K\vec{b}$

(ii) $(h+K)\vec{a} = h\vec{a} + K\vec{a}$, (Distributive law)

(iii) $(hK)\vec{a} = h(K\vec{a})$, (Associative law)

(iv) $1.\vec{a} = \vec{a}$

(v) $0.\vec{a} = \vec{0}$.

Position vector of a point

Let O be a fixed point called origin, let P be any other point, then the vector $\overrightarrow{OP}$ is called position vector of the point P relative to O and is denoted by $\vec{P}$.

As shown in figure - 13, let AB be any vector, then applying triangle law of addition we have $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ where $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ $\Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

Fig = 13

= (position vector of B) - (position vector of A)

Section Formula :- Let A and B be two points with position vector $\vec{a}$ and $\vec{b}$ respectively and P be a point on line segment AB, dividing it in the ration m:n internally. Then the position vector of P i.e. $\vec{r}$ is given by the formula :

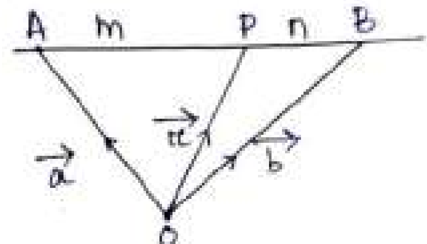


Fig - 14

$$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

If P divides AB externally in the ratio m:n then

$$\vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

If P is the midpoint of AB then $\vec{r} = \frac{\vec{a} + \vec{b}}{2}$

Example - 1 :- prove that by vector method the medians of a triangle are concurrent.

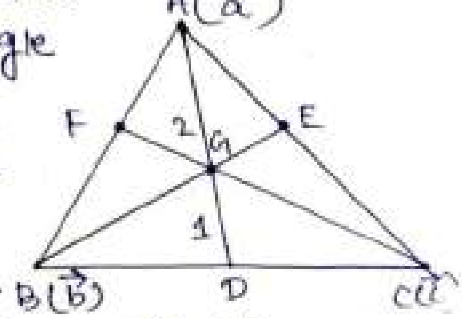


Fig - 15

Solution :- Let ABC be a triangle where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vector of A, B and C respectively. We have to show that the medians of this triangle are concurrent.

Let AD, BE and CF are the three medians of the triangle. Now as D be the midpoint of BC, so position vector of D i.e. $\vec{d} = \frac{\vec{b}+\vec{c}}{2}$.

Let G be any point of the median AD which divides AD in the ratio 2:1. Then position vector of G is given by $\vec{g} = \frac{2\vec{d}+\vec{a}}{2+1} = \frac{2\left(\frac{\vec{b}+\vec{c}}{2}\right)+1\vec{a}}{3}$

(by applying section formula)

$$\Rightarrow \vec{g} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

Let G' be point which divides BE in the ratio 2:1

Position vector of E is $\vec{e} = \frac{\vec{a}+\vec{c}}{2}$.

Then position vector of G' is given by $\vec{g'} = \frac{2\vec{e}+\vec{b}}{2+1} = \frac{2\left(\frac{\vec{a}+\vec{c}}{2}\right) + 1\vec{b}}{3}$

$\Rightarrow \vec{g'} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$ (by applying Section formula)

As position vector of a point is unique, so $G = G'$

Similarly if we take G'' be a point on CF dividing it in 2:1 ratio then the position vector of G'' will be same as that of G.

Hence G is the one point where three median meet.

$\therefore$ The three Medians of a triangle are concurrent. (proved)

Example 2: Prove that (i) $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$ (it is known as triangle inequality.

(ii) $|\vec{a}| - |\vec{b}| \le |\vec{a}-\vec{b}|$

(iii) $|\vec{a}-\vec{b}| \le |\vec{a}| + |\vec{b}|$

Proof:- Let O, A and B be three points, which are not collinear and then draw a triangle OAB.

Let $\vec{OA} = \vec{a}$, $\vec{AB} = \vec{b}$. then by triangle law of addition we have $\vec{OB} = \vec{a}+\vec{b}$

From properties of triangle we know that the sum of any two sides of a triangle is greater than the third side.

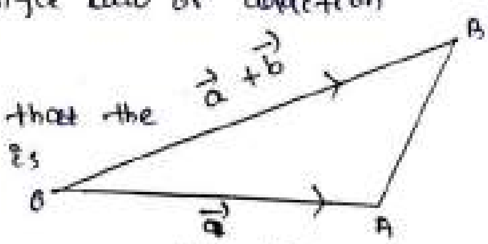


Fig - 16

$\Rightarrow OB < OA + AB$

$\Rightarrow |\vec{OB}| < |\vec{OA}| + |\vec{AB}|$

$\Rightarrow |\vec{a}+\vec{b}| < |\vec{a}| + |\vec{b}|$ ------- (1)

when O, A, B are colinear then from Fig-17 it is clear that $OB = OA + AB$

$\Rightarrow |\vec{OB}| = |\vec{OA}| + |\vec{AB}|$

$\Rightarrow |\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$ ------- (2)

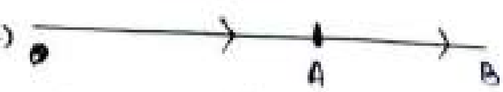


Fig-17

From (1) and (2) we have

$|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$ (proved)

(ii) $|\vec{a}| = |\vec{a}-\vec{b}+\vec{b}|$ ------- (1)

But $|(\vec{a}-\vec{b})+\vec{b}| \le |\vec{a}-\vec{b}|+|\vec{b}|$ (from triangle inequality) - (2)

From (1) and (2) we get $|\vec{a}| \le |\vec{a}-\vec{b}|+|\vec{b}|$

$\Rightarrow |\vec{a}|-|\vec{b}| \le |\vec{a}-\vec{b}|$ (proved)

(iii) $|\vec{a}-\vec{b}| = |\vec{a}+(-\vec{b})| \le |\vec{a}|+|-\vec{b}|$ (From triangle inequality)

$= |\vec{a}|+|\vec{b}|$ (as $|-\vec{b}| = |\vec{b}|$)

$|\vec{a}-\vec{b}| \le |\vec{a}|+|\vec{b}|$ (proved)

Components of vector in 2D

Let XOY be the co-ordinate plane and $P(x,y)$ be any point in this plane.

The unit vector along direction of X axis i.e $\vec{OX}$ is denoted by $\hat{i}$.

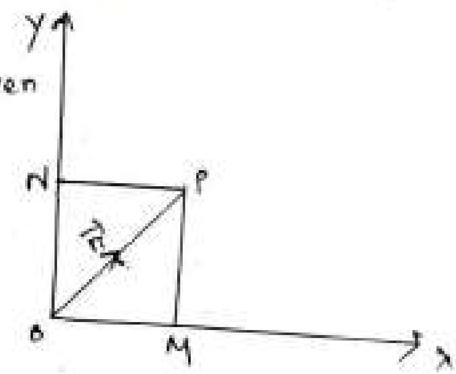
The unit vector along direction of Y axis i.e. $\vec{OY}$ is denoted by $\hat{j}$.

Then from figure-18 it is clear that $\vec{OM} = x\hat{i}$ and $\vec{ON} = y\hat{j}$

So, the position vector of P is given by

$$\boxed{\vec{OP} = \vec{r} = x\hat{i}+y\hat{j}}$$

And $OP = |\vec{OP}| = r = \sqrt{x^2+y^2}$



Representation of vector in component from in 2D

If $\vec{AB}$ is any vector having end points A(x_1,y_1) and B (x_2,y_2), then it can be represented by

$$\vec{AB} = (x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}$$

Components of vector in 3D

Let $P(x,y,z)$ be a point in space and $\hat{i}, \hat{j}$ and $\hat{k}$ be the unit vectors along x axis, y axis and z axis respectively. (as shown in fig 19)

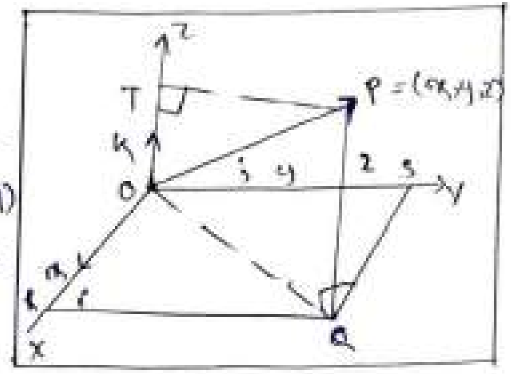


fig-19

Then the position vector of P is given by

$\vec{OP} = x\hat{i} + y\hat{j} + \hat{k}$, the vectors $x\hat{i}, y\hat{j}, z\hat{k}$ are called the components of $\vec{OP}$ along x-axis, y axis, and z-axis respectively.

$$\text{And } OP = |\vec{OP}| = \sqrt{x^2+y^2+z^2}$$

Addition and scalar Multiplication in terms of component form of vectors :-

For any vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

(i) $\vec{a} + \vec{b} = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j} + (a_3+b_3)\hat{k}$

(ii) $\vec{a} - \vec{b} = (a_1-b_1)\hat{i} + (a_2-b_2)\hat{j} + (a_3-b_3)\hat{k}$

(iii) $k\vec{a} = ka_1\hat{i} + ka_2\hat{j} + ka_3\hat{k}$, where k is a scalar.

(iv) $\vec{a} = \vec{b} \Leftrightarrow a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\Leftrightarrow a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$

Representation of vector in component from in 3-D and Distance between two points:

If $\vec{AB}$ is any vector having and end points $A(x_1, y_1 z_1)$ and $B(x_2, y_2 z_2)$, then it can be represented by position vector of B - position vector of A

$$\vec{AB} = \text{Position vector of B} - \text{position vector of A}$$
$$= (x_2\hat{i} - y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$
$$= (x_2-x_1)\hat{i} + (y_2-y_1)\hat{j} + (z_2-z_1)\hat{k}$$

$$|\vec{AB}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Example 3:-

Show that the points $A(2,6,3)$, $B(1,2,7)$ and $C(3,10,-1)$ are collinear.

Solution:- From given data position vector of A, $\vec{OA} = 2\hat{i} + 6\hat{j} + 3\hat{k}$.

Position vector of B, $\vec{OB} = \hat{i} + 2\hat{j} + 7\hat{k}$

Position vector of C, $\vec{OC} = 3\hat{i} + 10\hat{j} - \hat{k}$

Now $\vec{AB} = \vec{OB} - \vec{OA} = (1-2)\hat{i} + (2-6)\hat{j} + (7-3)\hat{k} = -\hat{i} - 4\hat{j} + 4\hat{k}$

Condition of perpendicularity :-

$\vec{AC} = \vec{OC} - \vec{OA} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$
$= -(-\hat{i} - 4\hat{j} + 4\hat{k}) = -\vec{AB}$

$\Rightarrow \vec{AB} \parallel \vec{AC}$ or collinear.

$\therefore$ They have same support and common point A.
As 'A' is common to both vector, that prove A, B and C are collinear.

Example-4 - prove that the points having position vector given by $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ form a right angled triangle. [2009(W)]

Solution:- Let A, B and C be the vertices of a triangle with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively.

Then. $\vec{AB}$ = position vector of B - position vector of A
$= (1-2)\hat{i} + (-3-(-1))\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\vec{BC}$ = position vector of C - position vector of B.
$= (3-1)\hat{i} + (-4-(-3))\hat{j} + (-4-(-5))\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{AC}$ = position vector of C - position vector of A.
$= (3-2)\hat{i} + (-4-(-1))\hat{j} + (-4-1)\hat{k} = \hat{i} - 3\hat{j} - 5\hat{k}$

Now $AB = |\vec{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$
$BC = |\vec{BC}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = 6$
$AC = |\vec{AC}| = \sqrt{1^2 + (-3)^2 + (-5)^2} = \sqrt{1+9+25} = \sqrt{35}$

from above $BC^2 + AC^2 = 6 + 35 = 41 = AB^2$
Hence ABC is right angled triangle.

Example-5:- Find the unit vector in the direction of the vector $\vec{a} = 3\hat{i} - 4\hat{j} + \hat{k}$. (2017-W)

Ans:- The unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3\hat{i} - 4\hat{j} + \hat{k}}{\sqrt{3^2 + (-4)^2 + 1^2}} = \frac{3\hat{i} - 4\hat{j} + \hat{k}}{9+16+1} = \frac{3}{\sqrt{26}}\hat{i} + \frac{4}{\sqrt{26}}\hat{j} + \frac{1}{\sqrt{26}}\hat{k}$$

Example-6 Find a unit vector in the direction of $\vec{a} + \vec{b}$ where $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 3\hat{k}$.

Ans:- Let $\vec{r} = \vec{a} + \vec{b} = (\hat{i} + \hat{j} - \hat{k}) + \hat{i} - \hat{j} + 3\hat{k} = 2\hat{i} + 2\hat{k}$
unit vector along direction of $\vec{a} + \vec{b}$ is given by

$$= \frac{\vec{r}}{|\vec{r}|} = \frac{2\hat{i} + 2\hat{k}}{\sqrt{2^2 + 2^2}} = \frac{2\hat{i} + 2\hat{k}}{\sqrt{8}} = \frac{2}{\sqrt{8}}\hat{i} + \frac{2}{\sqrt{8}}\hat{k}$$

$$= \frac{2}{2\sqrt{2}}\hat{i} + \frac{2}{2\sqrt{2}}\hat{j} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}.$$

Angle between the vectors:

As shown in figure-20 angle between two vectors $\overrightarrow{RS}$ and $\overrightarrow{PQ}$ can be determined as follows.

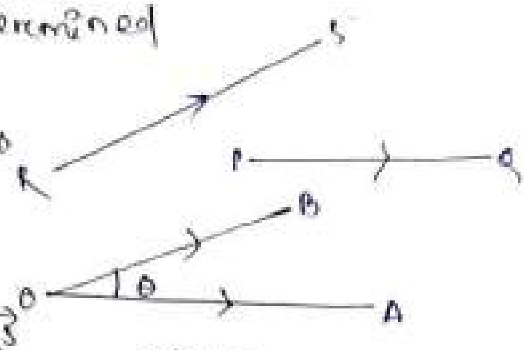


Fig-20

Let $\overrightarrow{OB}$ be a vector parallel to $\overrightarrow{RS}$ and $\overrightarrow{OA}$ is a vector parallel to $\overrightarrow{PQ}$ such that $\overrightarrow{OB}$ and $\overrightarrow{OA}$ intersect each other.

Then $\theta = \angle AOB$ = angle between $\overrightarrow{RS}$ and $\overrightarrow{PQ}$.

If $\theta = 0$ then vectors are said to be parallel.

If $\theta = \frac{\pi}{2}$ then vectors are said to be orthogonal or perpendicular.

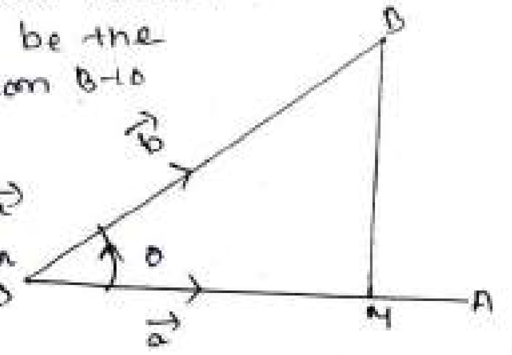
Dot product or scalar product of vectors

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ whose magnitudes are a and b respectively denoted by $\vec{a}\cdot\vec{b}$ is defined as the scalar $ab\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \le \theta \le \pi$

$$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = ab\cos\theta}$$

Geometrical meaning of dot product

In figure 21(a), $\vec{a}$ and $\vec{b}$ are two vectors having θ angle between them. Let M be the foot of the perpendicular drawn from B to OA.



Then OM is the projection of $\vec{b}$ on $\vec{a}$ and from figure-21(a) it is clear that,

$$|OM| = |OB|\cos\theta = |\vec{b}|\cos\theta.$$

Now $\vec{a}\cdot\vec{b} = |\vec{a}|\,(|\vec{b}|\cos\theta) = |\vec{a}| \times$ projection of $\vec{b}$ on $\vec{a}$

which gives projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Similarly we can write $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{b}|\,(|\vec{a}|\cos\theta) = |\vec{b}|$ projection of $\vec{a}$ on $\vec{b}$

Similarly, let us draw a perpendicular from A on OB and let N be the foot of the perpendicular in Fig-21(b).

Then ON = projection of $\vec{a}$ on $\vec{b}$

and ON = OA $\cos\theta = |\vec{a}|\cos\theta$

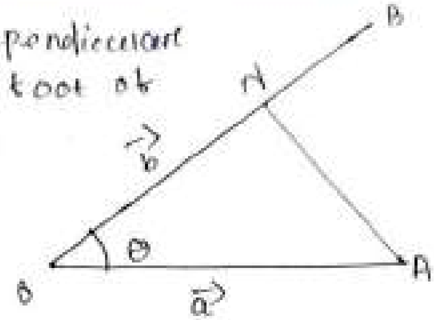


Fig-21(b)

<u>Properties of Dot product</u>

(i) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (communicative)

(ii) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (Distributive)

(iii) If $\vec{a} \parallel \vec{b}$, then $\vec{a}\cdot\vec{b} = ab$ {as $\theta = 0$ in this case $\cos 0 = 1$}

in particular $(\vec{a})^2 = \vec{a}\cdot\vec{a} = |\vec{a}|^2$

$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

(iv) If $\vec{a} \perp \vec{b}$ then $\vec{a}\cdot\vec{b} = 0$. {as $\theta = 90^\circ$ in this case $\cos 90^\circ = 0$}

In particular $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 = \hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{j} = \hat{i}\cdot\hat{k}$

(v) $\vec{a}\cdot\vec{0} = \vec{0}\cdot\vec{a} = 0$

(vi) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 = a^2 - b^2$ {where $|\vec{a}| = a$ and $|\vec{b}| = b$}

(vii) Work done by force:- The work done by a force $\vec{F}$ acting on a body causing displacement $\vec{d}$ is given by $W = \vec{F}\cdot\vec{d}$.

<u>Dot product in terms of rectangular components</u>

For any vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

we have,

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ (by applying distributive (ii), (iii) and (iv) successively)

<u>Angle between two non zero vectors</u>

For any two non zero vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, having θ is the angle between them we have,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab} = \hat{a}\cdot\hat{b} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

(in terms of components)

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

Condition of perpendicularity :

Two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are perpendicular to each other iff $a_1b_1 + a_2b_2 + a_3b_3 = 0$

Condition of parallelism :

Two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are parallel to each other iff $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

Scalar & vector projections of two vectors (Important formulas)

Scalar projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Vector projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\hat{a} = \left[\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\right]\vec{a}$

Scalar projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Vector projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\hat{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$

Examples :—

Q.7 Find the value of p for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$, $\hat{i} + p\hat{j} + 3\hat{k}$ are perpendicular to each other.

Solution :— Let $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$.

Here $a_1 = 3, a_2 = 2, a_3 = 9$

$b_1 = 1, b_2 = p$ & $b_3 = 3$

Given $\vec{a} \perp \vec{b} \Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

$\Rightarrow 3.1 + 2.p + 9.3 = 0$

$\Rightarrow 3 + 2p + 27 = 0$

$\Rightarrow 3 + 2p + 27 = 0$

$\Rightarrow 2p = -30$

$\Rightarrow p = -15$ (Ans)

Q.8 Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$, $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel to each other. (2014-W)

Solution :- Given $\vec{a} \parallel \vec{b} \leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \leftrightarrow \frac{3}{1} = \frac{2}{p} = \frac{9}{3}$

{Taking 1st two terms}

$\leftrightarrow 3 = \frac{2}{p} \leftrightarrow p = \frac{2}{3}$ (Ans) {Note - any two expression may be taken for finding p.}

Q.9 Find the scalar product of $3\hat{i} - 4\hat{j}$ and $-2\hat{i} + \hat{j}$. (2015-S)

Solution :- $(3\hat{i} - 4\hat{j}) \cdot (-2\hat{i} + \hat{j}) = (3 \times (-2)) + ((-4) \times 1)$
$= (-6) + (-4) = -10$

Q.10 Find the angle between the vectors $5\hat{i} + 3\hat{j} + 4\hat{k}$ and $6\hat{i} - 8\hat{j} - \hat{k}$. (2015-W)

Solution :- Let $\vec{a} = 5\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = 6\hat{i} - 8\hat{j} - \hat{k}$

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then $\theta = \cos^{-1}\left(\frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$

$= \cos^{-1}\left(\frac{5 \cdot 6 + 3 \cdot (-8) + 4(-1)}{\sqrt{5^2 + 3^2 + 4^2}\sqrt{6^2 + (-8)^2 + (-1)^2}}\right) = \cos^{-1}\left(\frac{30 - 24 - 4}{\sqrt{50}\sqrt{101}}\right)$

Q11. Find the Scalar and vector projection of $\vec{a}$ on $\vec{b}$ where $\vec{a} = \hat{i} - \hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} + 3\hat{k}$ (2013-W, 2017-W)

Solution:- Scalar Projection of $\vec{a}$ on $\vec{b}$ =

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{1\cdot 3 + (-1)\cdot 1 + (-1)\cdot 3}{\sqrt{3^2+1^2+3^2}} = \frac{3-1-3}{\sqrt{19}} = \frac{-1}{\sqrt{19}}$$

Vector projection of $\vec{a}$ on $\vec{b}$.

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = \frac{1\cdot 3 + (-1)\cdot 1 + (-1)\cdot 3}{(\sqrt{3^2+1^2+3^2})^2}(3\hat{i}+\hat{j}+3\hat{k})$$

$$= \frac{3-1-3}{19}(3\hat{i}+\hat{j}+3\hat{k}) = \frac{-1}{19}(3\hat{i}+\hat{j}+3\hat{k})$$

Q12. Find the Scalar and vector projection of $\vec{b}$ on $\vec{a}$ where $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ {2015-S}

Solution:- Scalar Projection of $\vec{b}$ on $\vec{a}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{3\cdot 2 + 1\cdot 3 + (-2)\cdot(-4)}{(\sqrt{3^2+1^2+(-2)^2})^2}(3\hat{i}+\hat{j}-2\hat{k})$$

$$= \frac{17}{14}(3\hat{i}+\hat{j}-2\hat{k})$$

Q-13 If $\vec{a}.\vec{b} = \vec{a}.\vec{c}$, then prove that $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b}-\vec{c})$ proof :- Given $\vec{a}.\vec{b} = \vec{a}.\vec{c}$

$$\Rightarrow (\vec{a}.\vec{b}) - (\vec{a}.\vec{c}) = \vec{0}$$

$$\Rightarrow \vec{a}.(\vec{b}-\vec{c}) = \vec{0}$$ (applying distributive property)

Dot product of above two vector is zero indicates the following conditions

$\vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{0}$ or $\vec{a} \perp (\vec{b}-\vec{c})$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b}-\vec{c})$ (proved)

Example-14 Find the work done by force $\vec{F} = \hat{i}+\hat{j}-\hat{k}$. acting on a particle if the particle is displace A from A(3,3,3) to B(4,4,4)

Ans :- Let O be the origin, then

position vector of A $\vec{OA} = 3\hat{i}+3\hat{j}+3\hat{k}$

position vector of B $\vec{OB} = 4\hat{i}+4\hat{j}+4\hat{k}$

Limit and continuity

In Mathematics, Differential calculus is a subfield of calculus that studies the ratio at which Quantities change.

Quantity

A quantity is an amount, number or Measurement, in other words an expression having value considered as a whole. These numbers can be expressed as a whole number, fraction, Decimals, Percentages.

Types of Quantities :

Quantities are broadly divided in to two categories.

(1). Constant

(2). variable

Constant

Constant is a symbol which remains the same value through out a set of mathematical operation. There are mainly two types of constant.

(i) absolute constant.

(ii). arbitary constant.

(i). Absolute Constant :-

Constants do not change whatever operation we may perform are known as absolute constant.

(ii) Arbitary Constant :-

In the equation $y = ax+b$ of a straight line are arbitary constants as they remain fixed for a particular straight line but vary from straight line where a & b are arbitrary constants.

(2) variable

variable is a symbol which can take various numerical values. Variables are of two types.

(i). Dependent variable

(ii). Independent variable

Ex: $y = 2x+4$

when $x=1$ then $y=6$, The value of y keeps on changing dependently upon the value of x. So x is the independent variable and y is the dependent variable.

Set :

A set is the collection of any well defined objects known as elements or numbers of the set.

Ex: $A = \{1, 2, 3, 4, 5\}$

Relation :

A Relation between two sets is a collection of ordered pairs containing one object from each sets. If the object x is from 1st set and the object y is from 2nd set then the objects are said to be related if the ordered pair (x, y) is in the relation.

Ex:

$A = \{1, 2\}$

$B = \{1, 2, 3\}$

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

↳ cartesian product.

(i). find a relation where
$x = y \rightarrow R_1 = \{(1,1), (2,2)\}$

ii) find a Relation where $x = y^2$
$\rightarrow R_2 = \{(1,1)\}$

(iii) find a relation where $\rightarrow x^2 = y \rightarrow R_3 = \{(1,1)\}$

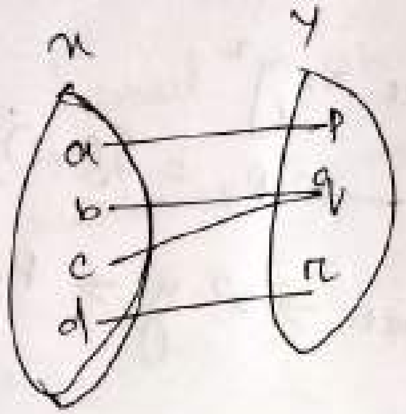
functions :-

A function is a special case of relation. Let (x, y) be two nonempty sets and R be a relation from x to y then R may not relate an element of x to an element of y or it may relate an element of x to more than an element of y.

But a function relates each element of x to an unique element of y

$f : x \rightarrow y$ (reads as "f maps x onto y")

Picturically a function Can be represented as



Main features of function :—

(i). To each element $x \in X$, there exists a unique element $y \in Y$ such that

$y = f(x)$

(ii) Distinct elements of X may be associated with the same elements of Y.

(iii) There may be elements of Y which are not associated with any element of X.

Domain

The set X is called the Domain of the function f.

Range

The set of all images of the elements of X under the mapping f is called the range of f is denoted by $f(x)$.

in general $f(x) \subseteq Y$.

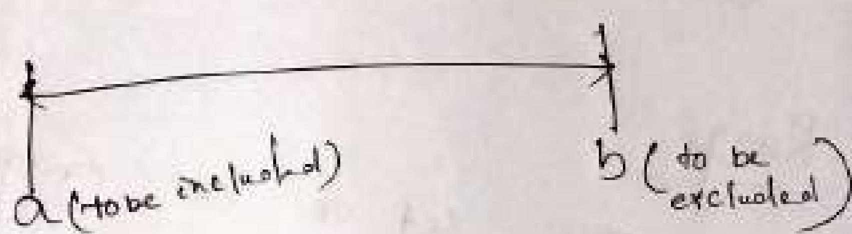
Intervals:

Let 'a' & 'b' two distinct real numbers $a \neq b$ and set $a < b$.

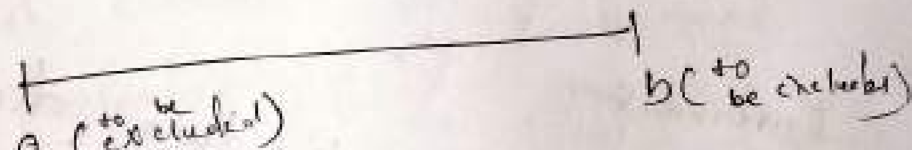
Then (i). $[a,b] = \{x \in R ; a \leq x \leq b\}$ is called Closed interval from a to b including a & b.

(ii) $(a,b) = \{x \in R : a < x < b\}$ is called open interval from a to b excluding a & b.

(iii). $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$ is called Semi closed and Semi open interval.



(iv) $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$ is called Semi open and semi closed interval.



Notes : ∞ will always comes with open Brackett, because we can't show ∞ on a number line. (as it is undefined)

(v). $[a,\infty) \rightarrow \{x \in \mathbb{R} : x \geq a\}$

(vi). $(a,\infty) \rightarrow \{x \in \mathbb{R} : x > a\}$

(vii). $(-\infty,a) \rightarrow \{x \in \mathbb{R} : x < a\}$

(viii) $(-\infty,a] \rightarrow \{x \in \mathbb{R} : x \leq a\}$

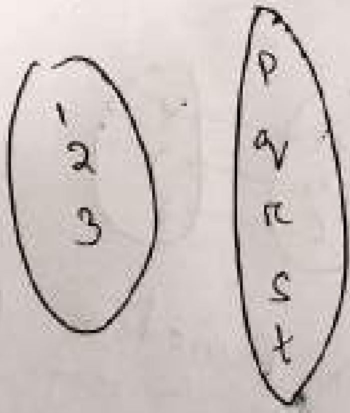
(ix) $(-\infty,\infty) \rightarrow \{x \in \mathbb{R}\}$

Classification Of function

A function f defined from the Set x to the set Y is said to be on to function of range as Fn, a Proper Subset of f in Y.

on

The function $f: x \to y$ is called on to function if there exist atleast one element of y which does not correspond to any element of x.



on to function (subjective mapping):—

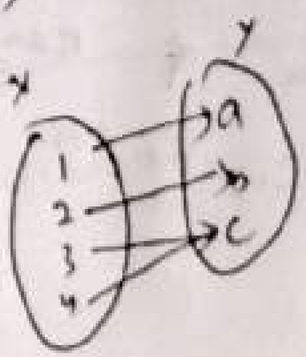
A function $f: x \to y$ is said to be an on to function if every element y is the image of same element in x.

If $X = \{1, 2, 3, 4\}$

$Y = \{a, b, c\}$

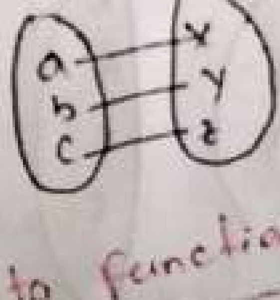
$F = \{(1,a), (2,b), (3,c), (4,c)\}$

(y set is Complety used)



One one mapping :-

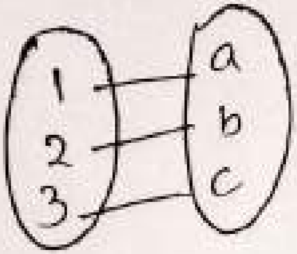
If distinct element of x have distinct image in y then the function called one one function.



One one and onto function (Bijection)

A function which is such that it is (i) onto (ii) one one is called Bijection.

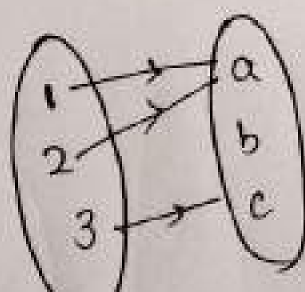
$X = \{1, 2, 3\}$; $Y = \{a, b, c\}$



(one one & onto)

Many-one function :-

A Function f form the Set X to the set Y is said to be Many one if there exists at least one element in Y which has more than one Preimage in X.



⇒ Types of function :-

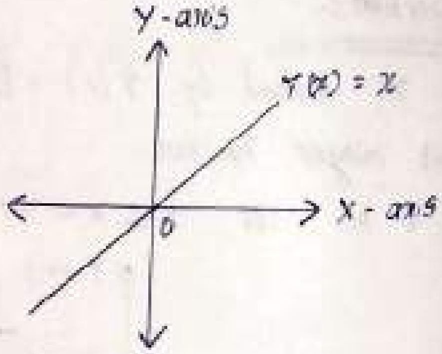
① Identity function :-

Defⁿ :- The function f defined by $f(x) = x$ for $\forall x \in R$ is called the identity function.

Domain = R

Range = R

Graph :-



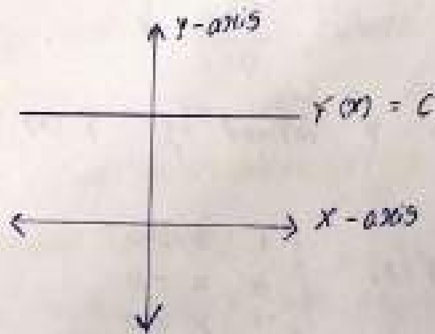
② Constant function :-

Defⁿ :- The function f defined by $f(x) = \frac{1}{x}$ is called Reciprocal function.

Domain = R

Range = C

Graph :-



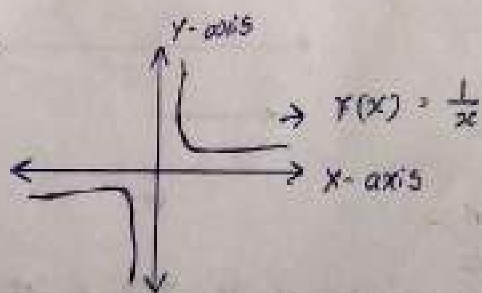
③ Reciprocal function :-

Defⁿ :- The function f defined by $f(x) = \frac{1}{x}$ is called Reciprocal function.

Domain = $R - \{0\}$

Range = $R - \{0\}$

Graph :-



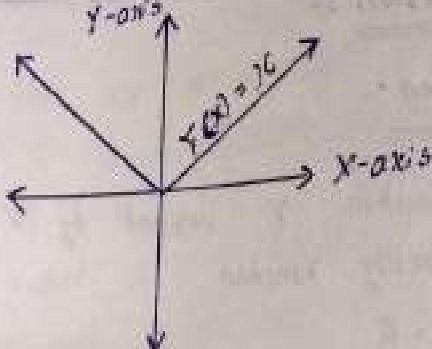
④ Modulus function :-

Defⁿ :- The function f defined by $f(x)$.

$$f(x) = |x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$ is called the modulus or absolute value function.

Domain = R , Range = $[0, \infty)$

Graph :-



⑤ Greatest integer function :-

Def^n :- The function f defined by $f(x) = [x]$ for all $x \in R$ is called the greatest integer function.

where $[x] = k$ iff $k \leq x < k+1$

$[n^+] = n$

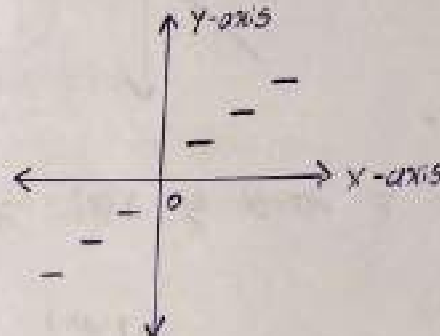
$[n^-] = n-1$

$[n] = n$

Domain = R

Range = $\mathbb{Z}$

Graph :-



⑥ Signum function :-

Def^n :- The function f defined by $f(x) = \begin{cases} \frac{|x|}{x} , & x \neq 0 \\ 0 , & \text{when } x = 0 \end{cases}$

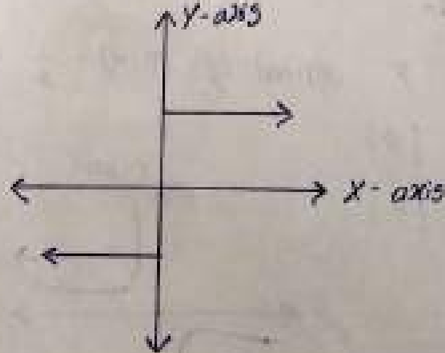
in other words

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Domain = R

Range = $\{1, 0, -1\}$

Graph :-

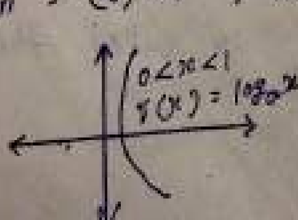


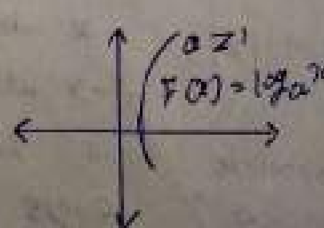
⑦ Logarithmic function :-

Def^n :- The function f defined by $f(x) = \log_a x$ where $x > 0$, $a > 0$ & $a \in R$ is called the Logarithmic function.

Domain = $(0, \infty)$, Range = R

Graph :-





Properties of Logarithmic Function :-

(i) $\log mn = \log m + \log n$

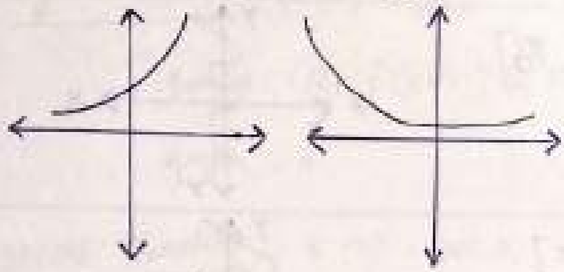
(ii) $\log m^n = n \log m$

(iii) $\log \frac{m}{n} = \log m - \log n$

Exponential Function :-

The function f defined by $f(x) = a^x$, where $a > 0$ and $x \in R$ is called exponential function.

$D = R$, $R = (0, \infty)$



Rational Function :-

The function f defined by $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomial functions and $h(x) \neq 0$ is called a Rational Function.

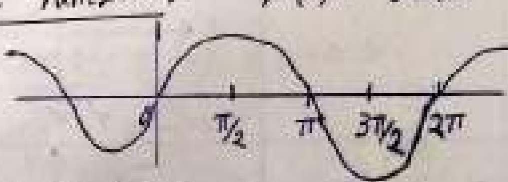
Domain :- Real No set except those values of x for which $h(x) = 0$

Trigonometric Function :- (circular function)

$\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$ are trigonometric functions.

(i) Sine function :- $f(x) = \sin x$ Domain $= R$

Range $= [-1, 1]$



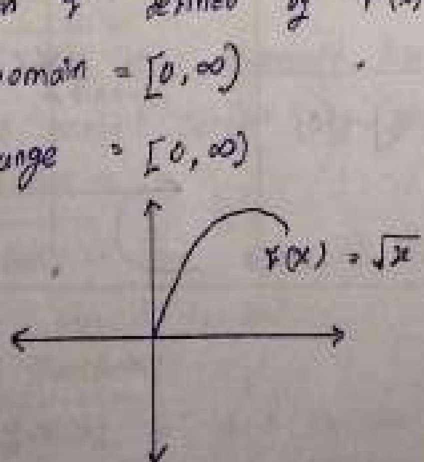
square root function :-

The function f defined by $f(x) = \sqrt{x}$ is called the square root function.

Domain $= [0, \infty)$

Range $= [0, \infty)$

Graph :-

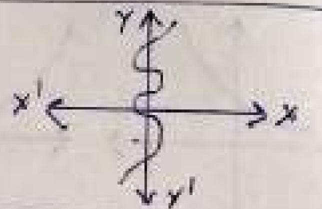
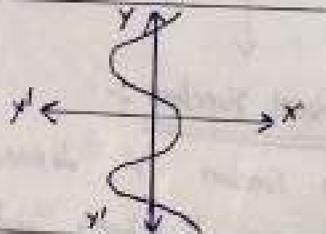
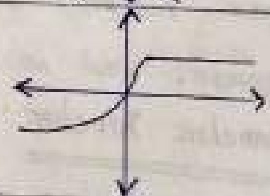
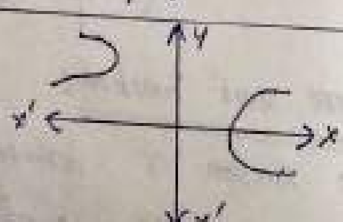
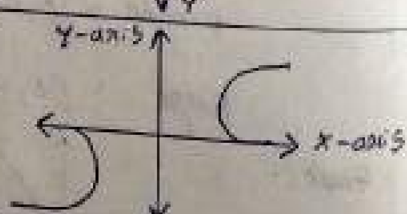


polynomial function :-

The function f defined by $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$, $a_n \neq 0$, where $a_0, a_1, a_2 \dots a_n$ are real nos and $n \in N$ is called a polynomial function of degree n.

Domain = R , Range = R

> Inverse Trigonometry :-

Function Name	Domain	Range	Graph
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$	
$\tan^{-1}x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$	
$\cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$	
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \{\pi/2\}$	
$\operatorname{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, \pi/2] - \{0\}$	

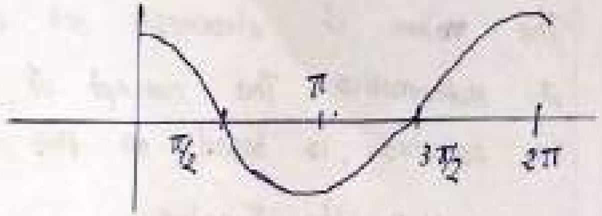
Even Function :- A Function $f(x)$ is Said to be an even Function if $f(-x) = f x$

odd Function :- A Function $f(x)$ is said to be an odd Function if $f(-x) = -f x$

cosine Function :- $f(x) = \cos x$

Domain $= R$

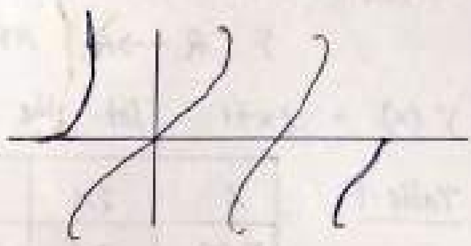
Range $= [-1, 1]$



Tangent Function :- $f(x) = \tan x$

Domain $= R - \{(2k+1)\pi/2 : k \in Z\}$

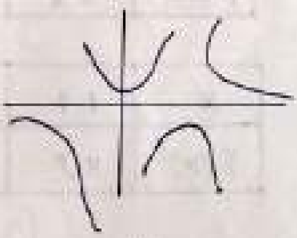
Range $= R$



secant Function :- $f(x) = \sec x$

Domain $= R - \{(2n+1)\pi/2 : n \in Z\}$

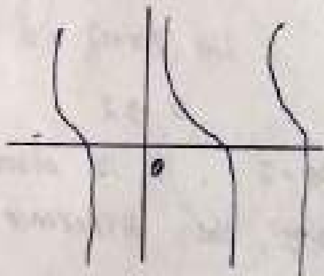
Range $= R - (-1, 1)$



cotangent Function :- $f(x) = \cot x$

Domain $= R - n\pi \quad n \in Z$

Range $= R$



cosecant Function :- $f(x) = \operatorname{cosec} x$

Domain $= R - n\pi$

Range $= R - (-1, 1)$

Explicitly Function :-

A Function which is expressed directly in terms of independence variable is called an explicit Function. $y = x^2 - 2x + 5$

Implicity Function :-

If a Function is not expressed directly in terms of independent variable is called an implicit Function.

Single valued Function :-

A Function $y = f(x)$ is said to be a single valued Function if there is one and only value of y corresponding to each value of x.

periodic function :-

A function $f(x)$ is said to be a periodic function if there exists a positive real constant T such that $f(x+T) = f(x)$, for all $x \in$

Introduction to Limits :-

The notion of closeness and nearness is basic in several branches of Mathematics. The concept of limit of a function, which is fundamental in calculus, is based on this notion.

consider the function,

$f : R \rightarrow R$ defined by

$f(x) = 2x+1$. Let the variable x takes values closer and closer to 2.

Table-1

x	2.1	2.01	2.001	2.0001
$f(x)$	5.2	5.02	5.002	5.0002

Table-2

x	1.9	1.99	1.999	1.9999
$f(x)$	4.8	4.98	4.998	4.9998

symbolically,

$$\lim_{x \to 2} (2x+1) = 5$$

From table-2, it is observed that the difference between x and 2 is decreasing, the difference between $f(x)$ and 6 is also decreasing corrospondly.

Limit of a function :-

Defn :- Let $f(x)$ be a function defined in some neighbourhood of a, except possibly at a and L be a number, we say that Limit of $f(x)$ as x approaches a is 'L' written $\lim_{x \to a} f(x) = L$. If for any $\epsilon > 0$ however small there exists a, $\delta > 0$ such that

$$\boxed{|f(x) - L| < \epsilon \text{ , whenever } 0 < |x-a| < \delta}$$

* To each $\epsilon > 0$, there exists a positive no. δ such that,

when $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$

Limit

- $(\epsilon - \delta)$ method
- calculation of L.H.L and R.H.L (Exist or not limits)
- Evaluation of Limit
- continuity

(i) **$(\epsilon - \delta)$ method :-**

Ex-1 Show that $\lim_{x \to 1} (3x+2) = 5$

Solution - $\lim_{x \to a} f(x) = l$

$f(x) = 3x+2$, $l = 5$; $a = 1$

$|f(x) - l| < \epsilon$

$|(3x+2) - 5| < \epsilon$

$|3x - 3| < \epsilon$

$3|x-1| < \epsilon$

$|x-1| < \epsilon/3$

$\boxed{|x-1| < \delta}$ $\left(\because \text{choose } \frac{\epsilon}{3} = \delta\right)$

so, $|(3x+2) - 5| < \epsilon \Rightarrow |x-1| < \delta$

Hence $\lim_{x \to 3} (3x+2) = 5$

Evaluation of Left hand and Right hand Limit :-

Left hand Limit :- To evaluate L.H.L of $f(x)$ at $x = a$

$$\lim_{x \to a^-} f(x)$$

Step-1 :- write $\lim_{x \to a^-} f(x)$

Step-2 :- Put $x = a-h$ and replace $x \to a-h$ by $h \to 0$ to obtain $\lim_{h \to 0} f(a-h)$

Step-3 :- simplify $\lim_{h \to 0} f(a-h)$ by using the

ϵ - δ method :-

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad |x-a| < \delta$$

→ prove that $\lim_{x \to 2} (2x+1) = 5$ using the ϵ - δ method / definition method :

Ans. $f(x) = 2x+1$, $L = 5$, $a = 2$

using the definition of limit $|f(x) - L| < \epsilon$

$$|(2x+1) - 5| < \epsilon$$

$$\Rightarrow |2x-4| < \epsilon$$

$$\Rightarrow |2(x-2)| < \epsilon$$

$$\Rightarrow |x-2| < \epsilon/2$$

$$\Rightarrow |x-2| < \delta \quad (\text{let, } \epsilon/2 = \delta)$$

So, $|(2x+1) - 5| < \epsilon = |x-2| < \delta$ (proved)

→ $\lim_{x \to 3} (3x+4) = 13$

Ans. $f(x) = 3x+4$, $L = 13$, $a = 3$

using the definition of limit $|f(x) - L| < \epsilon$

$$|(3x+4) - 13| < \epsilon$$

$$\Rightarrow |3x-9| < \epsilon$$

$$\Rightarrow |3(x-3)| < \epsilon$$

$$\Rightarrow |x-3| < \epsilon/3$$

$$\Rightarrow |x-3| < \delta \quad (\text{let, } \epsilon/3 = \delta)$$

So, $|(3x+4) - 13| < \epsilon = |x-3| < \delta$ (proved)

→ $\lim_{x \to -1} (4x-5) = -9$

Ans $f(x) = 4x-5$, $L = -9$, $a = -1$

$$|(4x-5) - (-9)| < \epsilon$$

$$\Rightarrow |4x-5+9| < \epsilon$$

$\Rightarrow |4x+4| < \epsilon$

$\Rightarrow |4(x+1)| < \epsilon$

$\Rightarrow |x+1| < \epsilon/4$

$\Rightarrow |x+1| < \delta$ (Let, $\epsilon/4 = \delta$)

so, $|(4x-5)-(-9)| < \epsilon = |x-(-1)| < \delta$

$\Rightarrow |(4x-5)+9| < \epsilon = |x+1| < \delta$ (proved)

$\Rightarrow$ L·H·L and R·H·L

L·H·L

$\lim_{x \to c^-} f(x)$

R·H·L

$\lim_{x \to c^+} f(x)$

Ex-1

check the existence of the function $f(x) = 3x+4$ at $x=2$.

Ans:

L·H·L	R·H·L
$\lim_{x \to c^-} f(x)$ $= \lim_{x \to 2^-} (3x+4)$ $= \lim_{h \to 0} (3(2-h)+4)$ $= 10$	$\lim_{x \to c^+} f(x)$ $= \lim_{x \to 2^+} (3x+4)$ $= \lim_{h \to 0} (3(2+h)+4)$ $= 10$

$\therefore$ L·H·L = R·H·L the limiting value exists.

Ex-2

$f(x) = [x]$ at $x=3$

Ans

L·H·L	R·H·L
$\lim_{x \to c^-} f(x)$ $= \lim_{x \to 3^-} [x]$ $= \lim_{h \to 0} [3-h]$ $= 2$	$\lim_{x \to c^+} f(x)$ $= \lim_{x \to 3^+} [x]$ $= \lim_{h \to 0} [3+h]$ $= 3$

→ Evaluation of Limit :-

① Algebric problem

② Trigonometric problem

③ Exponential or logarithmic

→ Algebric method
- Direct substitution method
- Factorisation method
- Rationalisation method
- some standard formula
- when $x \to \infty$ (infinity)

Direct substitution method :-

① $\lim_{x \to 1} (2x+3) = 2+3 = 5$

② $\lim_{x \to 2} \left(\frac{4x+2}{3x-1}\right) = \frac{10}{5} = 2$

③ $\lim_{x \to 0} (4x^5 - 3x^3 + 2x^2 + 1) = 1$

④ $\lim_{x \to -1} (3x-2)(4x-3) = -5 \times -7 = 35$

Factorisation method :-

① $\lim_{x \to 2} \frac{x^2-4}{x-2}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$

$= \lim_{x \to 2} x+2 = 2+2 = 4$

② $\lim_{x \to 1} \frac{x^2-2x+1}{x-1}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 1} \frac{(x-1)^{\cancel{2}}}{\cancel{(x-1)}}$

$= \lim_{x \to 1} x-1 = 1-1 = 0$

③ $\lim_{x \to 0} \frac{x^2-x}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 0} \frac{x(x-1)}{x} = 0-1 = -1$

Rationalisation method :-

① $\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}$

$= \lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{(\sqrt{x+1})^2-(1)^2}$

$= \lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{x+\cancel{1}-\cancel{1}}$

$= \lim_{x\to 0} \frac{\cancel{x}(\sqrt{x+1}+1)}{\cancel{x}} = \sqrt{1}+1 = 1+1 = 2$

② $\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{(\sqrt{x^2+4}-2)(\sqrt{x^2+4}+2)}{x^2(\sqrt{x^2+4}+2)}$

$= \lim_{x\to 0} \frac{(\sqrt{x^2+4})^2-(2)^2}{x^2(\sqrt{x^2+4}+2)}$

$= \lim_{x\to 0} \frac{x^2+4-4}{x^2(\sqrt{x^2+4}+2)}$

$= \lim_{x\to 0} \frac{x^2}{x^2(\sqrt{x^2+4}+2)} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$

using standard formula :-

using standard formula = $\boxed{\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}}$

① $\lim_{x\to 4} \frac{x^2-16}{x-4}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 4} \frac{(x)^2-(4)^2}{x-4} = 2\times 4^{2-1} = 2\times 4 = 8$

② $\lim_{x\to 2} \frac{x^3-8}{x-2}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 2} \frac{(x)^3-(2)^3}{x-2} = 3\times 2^{3-1} = 3\times 2^2 = 12$

Rationalisation method :-

① $\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0} \frac{x\left(\sqrt{x+1}+1\right)}{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}$

$= \lim_{x\to 0} \frac{x\left(\sqrt{x+1}+1\right)}{\left(\sqrt{x+1}\right)^2-(1)^2}$

$= \lim_{x\to 0} \frac{x\left(\sqrt{x+1}+1\right)}{x+1-1}$

$= \lim_{x\to 0} \frac{x\left(\sqrt{x+1}+1\right)}{x} = \sqrt{1}+1 = 1+1 = 2$

② $\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0} \frac{\left(\sqrt{x^2+4}-2\right)\left(\sqrt{x^2+4}+2\right)}{x^2\left(\sqrt{x^2+4}+2\right)}$

$= \lim_{x\to 0} \frac{\left(\sqrt{x^2+4}\right)^2-(2)^2}{x^2\left(\sqrt{x^2+4}+2\right)}$

$= \lim_{x\to 0} \frac{x^2+4-4}{x^2\left(\sqrt{x^2+4}+2\right)}$

$= \lim_{x\to 0} \frac{x^2}{x^2\left(\sqrt{x^2+4}+2\right)} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$

using standard formula:-

using standard formula = $\boxed{\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}}$

① $\lim_{x\to 4} \frac{x^2-16}{x-4}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 4} \frac{(x)^2-(4)^2}{x-4} = 2\times 4^{2-1} = 2\times 4 = 8$

② $\lim_{x\to 2} \frac{x^3-8}{x-2}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 2} \frac{(x)^3-(2)^3}{x-2} = 3\times 2^{3-1} = 3\times 2^2 = 12$

(3) $\lim_{x \to 5} \frac{\sqrt{x} - \sqrt{5}}{x-5}$ ($\frac{0}{0}$ form)

$= \lim_{x \to 5} \frac{(x)^{1/2} - (5)^{1/2}}{x-5} = \frac{1}{2} \times 5^{\frac{1}{2}-1} = \frac{1}{2} \times 5^{-1/2} = \frac{1}{2} \times \frac{1}{\sqrt{5}} = \frac{1}{2\sqrt{5}}$

(4) $\lim_{x \to 1} \frac{x^5 - 1}{x-1}$ ($\frac{0}{0}$ form)

$= \lim_{x \to 1} \frac{(x)^5 - (1)^5}{x-1} = 5 \times 1^{5-4} = 5 \times 1 = 5$

(5) $\lim_{x \to -1} \frac{x^3 + 1}{x+1}$ ($\frac{0}{0}$ form)

$= \lim_{x \to -1} \frac{x^3 - (-1)}{x - (-1)}$

$= \lim_{x \to -1} \frac{(x)^3 - (-1)^3}{x - (-1)} = 3 \times (-1)^{3-1} = 3 \times (-1)^2 = 3$

(6) $\lim_{x \to 2} \frac{x^5 - 32}{x^3 - 8}$ ($\frac{0}{0}$ form)

$= \lim_{x \to 2} \frac{(x)^5 - (2)^5}{(x)^3 - (2)^3}$

Devide the numerator and denominator by $(x-2)$.

$= \lim_{x \to 2} \frac{\frac{(x)^5 - (2)^5}{x-2}}{\frac{(x)^3 - (2)^3}{x-2}}$

$= \frac{\lim_{x \to 2} \frac{(x)^5 - (2)^5}{x-2}}{\lim_{x \to 2} \frac{(x)^3 - (2)^3}{x-2}} = \frac{5 \times (2)^{5-1}}{3 \times (2)^{3-1}} = \frac{5 \times 2^4}{3 \times 2^2} = \frac{80}{12}$

(7) $\lim_{x \to 0} \frac{(3+2x)^3 - 27}{x}$ ($\frac{0}{0}$ form)

Ans $\lim_{x \to 0} \frac{(3+2x)^3 - (3)^3}{x}$

$2 \lim_{x \to 0} \frac{(3+2x)^3 - (3)^3}{3+2x-3}$

$2 \lim_{3+2x \to 3} \frac{(3+2x)^3 - (3)^3}{(3+2x) - 3} = 2 \times 3 \times (3)^{3-1} = 2 \times 3 \times 3^2 = 54$

⑧ $\lim_{x \to 0} \frac{(4-5x)^3 - 64}{x}$

Ans. $\lim_{x \to 0} \frac{(4-5x)^3 - (4)^3}{x}$

$-5 \lim_{x \to 0} \frac{(4-5x)^3 - (4)^3}{4-5x-4}$

$-5 \lim_{4-5x \to 4} \frac{(4-5x)^3 - (4)^3}{(4-5x)-4} = -5 \times 3 \times (4)^{3-1} = -5 \times 3 \times 4^2 = -240$

⑨ $\lim_{x \to 2} \frac{\frac{1}{x^2} - \frac{1}{4}}{x-2}$

Ans. $\lim_{x \to 2} \frac{(x)^{-2} - (2)^{-2}}{x-2}$

$= -2 \times (2)^{-2-1} = -2 \times (2)^{-3} = -2 \times \frac{1}{8} = -\frac{1}{4}$

⑩ $\lim_{x \to 2} \frac{x-2}{x^4 - 16}$

Ans. $\lim_{x \to 2} \frac{x-2}{(x)^4 - (2)^4}$

$\lim_{x \to 2} \frac{\frac{x-2}{x-2}}{\frac{(x)^4 - (2)^4}{x-2}}$

$\frac{\lim_{x \to 2} \frac{x-2}{x-2}}{\lim_{x \to 2} \frac{(x)^4 - (2)^4}{x-2}} = \frac{\cancel{1 \times (2)^{1-1}}\ 1}{(4) \times (2)^{4-1}} = \frac{1}{32}$

⑪ $\lim_{x \to 0} \frac{(4-x)^4 - 256}{x}$

Ans. $\lim_{x \to 0} \frac{(4-x)^4 - (4)^4}{x}$

$-1 \lim_{x \to 0} \frac{(4-x)^4 - (4)^4}{4-x-4}$

$-1 \lim_{4-x \to 4} \frac{(4-x)^4 - (4)^4}{(4-x)-4} = -1 \times 4 \times (4)^{4-1} = -1 \times 4 \times 4^3 = -256$

⇒ Method of evaluating when $x \to \infty$ (Infinity) :-

Formula ⇒ Take the highest power of x common from the numerator and denominator.

Ex ⇒

① $\lim_{x \to \infty} \frac{4x+1}{5x-1}$

Ans. $\lim_{x \to \infty} \frac{\cancel{x}\left(4+\frac{1}{x}\right)}{\cancel{x}\left(5-\frac{1}{x}\right)}$

$\lim_{x \to \infty} \frac{4+\frac{1}{x}}{5-\frac{1}{x}} = \frac{4+\frac{1}{\infty}}{5-\frac{1}{\infty}} = \frac{4+0}{5-0} = \frac{4}{5}$

② $\lim_{x \to \infty} \frac{3x^2+4x+5}{x^2+x}$

Ans. $\lim_{x \to \infty} \frac{\cancel{x^2}\left(3+\frac{4}{x}+\frac{5}{x^2}\right)}{\cancel{x^2}\left(1+\frac{1}{x}\right)}$

$\lim_{x \to \infty} \frac{3+\frac{4}{x}+\frac{5}{x^2}}{1+\frac{1}{x}} = \frac{3+0+0}{1+0} = \frac{3}{1} = 3$

③ $\lim_{x \to \infty} \frac{4x-2}{x^2+x+1}$

Ans. $\lim_{x \to \infty} \frac{\cancel{x}\left(4-\frac{2}{x}\right)}{x^{\cancel{2}}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}$

$\lim_{x \to \infty} \frac{4-\frac{2}{x}}{x\left(1+\frac{1}{x}+\frac{1}{x^2}\right)} = \frac{4-0}{\infty(1+0+0)} = \frac{4}{\infty} = 0$

④ $\lim_{x \to \infty} \frac{3x^2+x-1}{2x^2-7x+5}$

Ans. $\lim_{x \to \infty} \frac{\cancel{x^2}\left(3+\frac{1}{x}-\frac{1}{x^2}\right)}{\cancel{x^2}\left(2-\frac{7}{x}+\frac{5}{x^2}\right)}$

$= \lim_{x \to \infty} \frac{3+\frac{1}{x}-\frac{1}{x^2}}{2-\frac{7}{x}+\frac{5}{x^2}}$

$= \frac{3+0-0}{2-0+0} = \frac{3}{2}$

⑤ $\lim_{x\to\infty} \dfrac{1+2+3+ ---- +n}{n^2}$

Ans. $\lim_{x\to\infty} \dfrac{\frac{n(n+1)}{2}}{n^2}$

$\lim_{x\to\infty} \dfrac{n(n+1)}{2n^2}$

$\lim_{x\to\infty} \dfrac{n^2+n}{2n^2}$

$\lim_{x\to\infty} \dfrac{\cancel{n^2}(1+\frac{1}{n})}{2\cancel{n^2}} = \dfrac{1+\frac{1}{n}}{2} = \dfrac{1}{2}$

⑥ $\lim_{x\to\infty} \dfrac{1^2+2^2+3^2+ ----- +n^2}{n^3}$

Ans. $\lim_{x\to\infty} \dfrac{\frac{n(n+1)(2n+1)}{6}}{n^3}$

$\lim_{x\to\infty} \dfrac{(n^2+n)(2n+1)}{6n^3}$

$\lim_{x\to\infty} \dfrac{2n^3+n^2+2n^2+n}{6n^3}$

$\lim_{x\to\infty} \dfrac{2n^3+3n^2+n}{6n^3}$

$\lim_{x\to\infty} \dfrac{\cancel{n^3}(2+\frac{3}{n}+\frac{1}{n^2})}{6\cancel{n^3}}$

$\lim_{x\to\infty} \dfrac{2+\frac{3}{n}+\frac{1}{n^2}}{6} = \dfrac{2+0+0}{6}$

$= \dfrac{\cancel{2}}{\cancel{6}3}$

$= \dfrac{1}{3}$

⇒ Trigonometric Function :-

Indeterminate Form :- $\frac{0}{0}$, $\frac{\infty}{\infty}$

① $0 \times \infty$ ④ 1^{∞}
② $\infty - \infty$ ⑤ ∞^0
③ 0^0

① $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

② $\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1$

③ $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$

④ $\lim_{\theta \to 0} \frac{\theta}{\tan\theta} = 1$

EX→

① $\lim_{x \to 0} \frac{\sin 2x}{x}$

Ans. $\lim_{x \to 0} \frac{2\sin 2x}{2x}$

$2 \lim_{2x \to 0} \frac{\sin 2x}{2x} = 2 \times 1 = 2$

② $\lim_{x \to 0} \frac{\sin 5x}{x}$

Ans. $\lim_{x \to 0} \frac{5 \sin 5x}{5x}$

$5 \lim_{5x \to 0} \frac{\sin 5x}{5x} = 5 \times 1 = 5$

③ $\lim_{x \to 0} \frac{x}{\sin 10x}$

Ans. $\lim_{x \to 0} \frac{10x}{10 \sin 10x}$

$\frac{1}{10} \lim_{x \to 0} \frac{10x}{\sin 10x}$

$\frac{1}{10} \lim_{10x \to 0} \frac{10x}{\sin 10x} = \frac{1}{10} \times 1 = \frac{1}{10}$

④ $\lim_{x \to 0} \frac{\tan 3x}{x}$

5. $\lim_{x \to 0} \frac{3\tan 3x}{3x}$

$3 \lim_{3x \to 0} \frac{\tan 3x}{3x} = 3 \times 1 = 3$

⑤ $\lim_{x \to 0} \frac{\sin x/2}{x}$

Ans. $\lim_{x \to 0} \frac{\frac{1}{2}\sin x/2}{\frac{1}{2}x}$

$\frac{1}{2} \lim_{\frac{x}{2} \to 0} \frac{\sin x/2}{x/2} = \frac{1}{2} \times 1 = \frac{1}{2}$

⑥ $\lim_{x \to 0} \frac{\tan(-4x)}{x}$

Ans. $\lim_{x \to 0} \frac{-4 \tan(-4x)}{-4x}$

$-4 \lim_{-4x \to 0} \frac{\tan(-4x)}{-4x} = -4 \times 1 = -4$

⑦ $\lim_{x \to 0} \frac{\tan 100x}{x}$

Ans. $\lim_{x \to 0} \frac{100 \tan 100x}{100x}$

$100 \lim_{100x \to 0} \frac{\tan 100x}{100x} = 100 \times 1 = 100$

⑧ $\lim_{x \to 0} \frac{\sin 5x}{4x}$

Ans. $\lim_{x \to 0} \frac{\frac{1}{4}\sin 5x}{\frac{1}{4} \times 4x}$

$\frac{1}{4} \lim_{x \to 0} \frac{\sin 5x}{x}$

$\frac{1}{4} \lim_{x \to 0} \frac{5 \sin 5x}{5x}$

$= \frac{5}{4} \lim_{x \to 0} \frac{\sin 5x}{5x}$

$= \frac{5}{4} \lim_{5x \to 0} \frac{\sin 5x}{5x}$

$= \frac{5}{4} \times 1 = \frac{5}{4}$

⑨ $$\lim_{x\to 0} \frac{\sin 3x}{\sin 4x}$$

Ans $$\lim_{x\to 0} \frac{\frac{\sin 3x}{x}}{\frac{\sin 4x}{x}}$$

$$\lim_{x\to 0} \frac{\frac{3\sin 3x}{3x}}{\frac{4\sin 4x}{4x}}$$

$$\frac{3 \lim\limits_{3x\to 0} \frac{\sin 3x}{3x}}{4 \lim\limits_{4x\to 0} \frac{\sin 4x}{4x}} = \frac{1\times 3}{1\times 4} = \frac{3}{4}$$

⑩ $$\lim_{x\to 0} \frac{\sin \alpha x}{\tan \beta x}$$

Ans. $$\lim_{x\to 0} \frac{\frac{\sin\alpha x}{x}}{\frac{\tan\beta x}{x}}$$

$$\lim_{x\to 0} \frac{\frac{\alpha \sin\alpha x}{\alpha x}}{\frac{\beta \tan\beta x}{\beta x}}$$

$$\frac{\alpha \lim\limits_{\alpha x\to 0} \frac{\sin \alpha x}{\alpha x}}{\beta \lim\limits_{\beta x\to 0} \frac{\tan\beta x}{\beta x}} = \frac{\alpha\times 1}{\beta\times 1} = \frac{\alpha}{\beta}$$

⑪ $$\lim_{x\to 0} \frac{\tan x/3}{\sin 4x/5}$$

Ans $$\lim_{x\to 0} \frac{\frac{\tan x/3}{x}}{\frac{\sin 4x/5}{x}}$$

$$\lim_{x\to 0} \frac{\frac{1/3 \tan x/3}{x/3}}{\frac{4/5 \sin 4x/5}{4x/5}}$$

$$\frac{\frac{1}{3} \lim\limits_{\frac{1}{3}x\to 0} \frac{\tan x/3}{x/3}}{\frac{4}{5} \lim\limits_{\frac{4}{5}x\to 0} \frac{\sin 4x/5}{4x/5}} = \frac{\frac{1}{3}}{\frac{4}{5}} = \frac{1}{3}\times\frac{5}{4} = \frac{5}{12}$$

⑫ $$\lim_{x\to 0} \frac{\sin 4x}{x/2}$$

Ans. $$\lim_{x\to 0} \frac{8 \sin 4x}{8\times x/2}$$

$$8 \lim_{4x\to 0} \frac{\sin 4x}{4x} = 8\times 1 = 8$$

⑬ $$\lim_{x\to 0} \frac{\sin x^\circ}{x}$$

$180^\circ = \pi$

$1^\circ = \frac{\pi}{180}$

$x = \frac{\pi x}{180}$

Ans. $$\lim_{x\to 0} \frac{\sin \pi x/180}{x}$$

$$\lim_{x\to 0} \frac{\pi/180 \sin \pi x/180}{\pi/180\, x}$$

$$\frac{\pi}{180} \lim_{\frac{\pi}{180}x\to 0} \frac{\sin \pi x/180}{\pi x/180} = \frac{\pi}{180}\times 1 = \frac{\pi}{180}$$

⑭ $$\lim_{x\to 0} \frac{\tan 3x}{\sin 4x}$$

Ans. $$\lim_{x\to 0} \frac{\frac{\tan 3x}{x}}{\frac{\sin 4x}{x}}$$

$$\lim_{x\to 0} \frac{\frac{3\tan 3x}{3x}}{\frac{4\sin 4x}{4x}}$$

$$\frac{3 \lim\limits_{3x\to 0} \frac{\tan 3x}{3x}}{4 \lim\limits_{4x\to 0} \frac{\sin 4x}{4x}} = \frac{3\times 1}{4\times 1} = \frac{3}{4}$$

(15) $\lim\limits_{h \to 0} \dfrac{\sin(x+h) - \sin x}{h}$

Ans. $\lim\limits_{h \to 0} \dfrac{2\cos\left(\frac{x+h+x}{2}\right) \sin\left(\frac{x+h-x}{2}\right)}{h}$

$\lim\limits_{h \to 0} \dfrac{2\cos\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right)}{h}$

$\lim\limits_{h \to 0} \cancel{2} \cos\left(\frac{2x+h}{2}\right) \times \dfrac{\sin h/2}{\cancel{2} \times h/2}$

$\lim\limits_{h \to 0} \cos\left(\frac{2x+h}{2}\right) \times \dfrac{\sin h/2}{h/2}$

$\lim\limits_{h \to 0} \cos\left(\frac{2x+h}{2}\right) \lim\limits_{\frac{h}{2} \to 0} \dfrac{\sin h/2}{h/2}$

$= \cos x \times 1 = \cos x$

(16) $\lim\limits_{h \to 0} \dfrac{\cos(x+h) - \cos x}{h}$

Ans. $\lim\limits_{h \to 0} \dfrac{2\sin\left(\frac{x+h+x}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{h}$

$\lim\limits_{h \to 0} \dfrac{2\sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{h}$

$-\lim\limits_{h \to 0} \dfrac{2\sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right)}{h}$

$-\lim\limits_{h \to 0} \cancel{2} \sin\left(\frac{2x+h}{2}\right) \dfrac{\sin h/2}{\cancel{2} \times h/2}$

$-\lim\limits_{h \to 0} \sin\left(\frac{2x+h}{2}\right) \lim\limits_{h/2 \to 0} \dfrac{\sin h/2}{h/2}$

$= -\sin x \times 1 = -\sin x$

(17) $\lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$

Ans. $\lim_{h \to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$

$\lim_{h \to 0} \frac{\frac{\sin(x+h).\cos x - \sin x \cdot \cos(x+h)}{\cos x \times \cos(x+h)}}{h}$

$\lim_{h \to 0} \frac{\frac{\sin(x+h-x)}{\cos x \times \cos(x+h)}}{h}$

$\lim_{h \to 0} \frac{\sin h}{\cos x \times \cos(x+h)} \times \frac{1}{h}$

$\lim_{h \to 0} \frac{\sin h}{h} \times \frac{1}{\cos x \times \cos(x+h)}$

$\lim_{h \to 0} \frac{\sin h}{h} \lim_{h \to 0} \frac{1}{\cos x \times \cos(x+h)}$

$= 1 \times \frac{1}{\cos x \times \cos x} = \frac{1}{\cos^2 x} = \sec^2 x$

(18) $\lim_{x \to 0} \frac{x}{\sin 2x}$

Ans. $\lim_{x \to 0} \frac{2x}{2 \sin 2x}$

$\frac{1}{2} \lim_{x \to 0} \frac{2x}{\sin 2x}$

$\frac{1}{2} \lim_{2x \to 0} \frac{2x}{\sin 2x} = \frac{1}{2} \times 1 = \frac{1}{2}$

(19) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

Ans $\lim_{x \to 0} \frac{2 \sin^2 x/2}{x^2}$

$\lim_{x \to 0} \cancel{2} \frac{\sin^2 x/2}{\cancel{2}^4 \times \frac{x^2}{4}}$

$\lim_{x \to 0} \frac{\sin^2 x/2}{2\, x^2/4}$

$= \frac{1}{2} \lim_{x \to 0} \left(\frac{\sin x/2}{x/2}\right)^2$

$= \frac{1}{2} \lim_{x/2 \to 0} \left(\frac{\sin x/2}{x/2}\right)^2$

$= \frac{1}{2} \times (1)^2 = \frac{1}{2}$

2nd method

$\lim_{x\to 0} \frac{1-\cos x}{x^2}$

$\lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}$

$\lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)}$

$\lim_{x\to 0} \frac{\sin^2 x}{x^2} \times \frac{1}{1+\cos x}$

$\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \lim_{x\to 0} \frac{1}{1+\cos x}$

$= (1)^2 \times \frac{1}{2} = \frac{1}{2}$

(20) $\lim_{x\to \pi/2} (\sec x - \tan x)$ ($\infty - \infty$ form)

Ans $\lim_{x\to \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$

$\lim_{x\to \pi/2} \left(\frac{1-\sin x}{\cos x}\right)$

$\lim_{x\to \pi/2} \frac{(1-\sin x)(1+\sin x)}{\cos x(1+\sin x)}$

$\lim_{x\to \pi/2} \frac{1-\sin^2 x}{\cos x(1+\sin x)}$

$\lim_{x\to \pi/2} \frac{\cos^2 x}{\cos x(1+\sin x)}$

$\lim_{x\to \pi/2} \frac{\cos x}{1+\sin x} = \frac{0}{2} = 0$

2nd method

$\lim_{x\to \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$

$\lim_{x\to \pi/2} \left(\frac{1-\sin x}{\cos x}\right)$

$\lim_{x\to \pi/2} \frac{\cos x(1-\sin x)}{\cos x(\cos x)}$

$\lim_{x\to \pi/2} \frac{\cos x(1-\sin x)}{\cos^2 x}$

$\lim_{x\to \pi/2} \frac{\cos x(1-\sin x)}{1-\sin^2 x}$

$\lim_{x\to \pi/2} \frac{\cos x\cancel{(1-\sin x)}}{\cancel{(1-\sin x)}(1+\sin x)}$

$\lim_{x\to \pi/2} \frac{\cos x}{1+\sin x} = \frac{0}{2} = 0$

(21) $\lim_{x\to \pi} \frac{1+\cos x}{\tan^2 x}$ ($\frac{0}{0}$ form)

Ans $\lim_{x\to \pi} \frac{1+\cos x}{\frac{\sin^2 x}{\cos^2 x}}$

$\lim_{x\to \pi} \frac{(1+\cos x)\cos^2 x}{\sin^2 x}$

$\lim_{x\to \pi} \frac{(1+\cos x)\cos^2 x}{1-\cos^2 x}$

$\lim_{x\to \pi} \frac{\cancel{(1+\cos x)}\cos^2 x}{\cancel{(1+\cos x)}(1-\cos x)}$

$\lim_{x\to \pi} \frac{\cos^2 x}{1-\cos x} = \frac{1}{2}$

2nd method

$\lim_{x\to \pi} \frac{(1+\cos x)\cos^2 x}{\sin^2 x}$

$\lim_{x\to \pi} \frac{(1+\cos x)(1-\cos x)\cos^2 x}{\sin^2 x(1-\cos x)}$

$\lim_{x\to \pi} \frac{(1-\cos^2 x)\cos^2 x}{\sin^2 x(1-\cos x)}$

$\lim_{x\to \pi} \frac{\cancel{\sin^2 x} \times \cos^2 x}{\cancel{\sin^2 x}(1-\cos x)}$

$\lim_{x\to \pi} \frac{\cos^2 x}{1-\cos x} = \frac{1}{2}$

⇒ Exponential / Logarithmic :-

→ $\lim_{x\to 0} (1+x)^{1/x} = e$

→ $\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^x = e$

→ $\lim_{x\to 0} \frac{a^x-1}{x} = \ln a$

→ $\lim_{x\to 0} \frac{e^x-1}{x} = \ln e = 1$

→ $\log mn = \log m + \log n$

→ $\log \frac{m}{n} = \log m - \log n$

→ $\log m^n = n \log m$

$\boxed{\ln = \log_e}$

→ $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

Q① $\lim_{x\to 0} \frac{a^x-1}{x}$

Ans. $\ln a$

② $\lim_{x\to 0} \frac{2^x-1}{x}$

Ans. $\ln 2$

③ $\lim_{x\to 0} \frac{2^{2x}-1}{x}$

Ans. 1st method

$2 \lim_{2x\to 0} \frac{2^{2x}-1}{2x}$

$= 2\ln 2$

2nd method

$\lim_{x\to 0} \frac{(2^2)^x-1}{x}$

$= \ln(2)^2$

$= 2\ln 2$

④ $\lim_{x\to 0} \frac{3^{2x}-1}{x}$

Ans. $\lim_{x\to 0} \frac{(3^2)^x-1}{x} = \ln(3)^2$

$= 2\ln 3$

⑤ $\lim_{x\to 0} \frac{4^{10x}-1}{x}$

Ans. $10 \lim_{10x\to 0} \frac{4^{10x}-1}{10x} = 10\ln 4$

⑥ $\lim_{x\to 0} \frac{2^{5x}-1}{x}$

Ans. $\lim_{x\to 0} \frac{(2^5)^x-1}{x} = \ln(2)^5$

$= 5\ln 2$

⑦ $\lim_{x\to 0} \frac{2^x-3^x}{x}$

Ans. $\lim_{x\to 0} \frac{2^x-1+1-3^x}{x}$

$\lim_{x\to 0} \frac{(2^x-1)-(3^x-1)}{x}$

$\lim_{x\to 0} \frac{2^x-1}{x} - \lim_{x\to 0} \frac{3^x-1}{x}$

$= \ln 2 - \ln 3$

⑧ $\lim_{x\to 1} \frac{2^{x-1}-1}{x-1}$ IMP

Ans. Let, $x-1 = y$

$\Rightarrow x = y+1$

$\lim_{y+1\to 1} \frac{2^y-1}{y}$

$\lim_{y\to 0} \frac{2^y-1}{y} = \ln 2$

⑨ $\lim_{x\to 0} \frac{a^x-b^x}{x}$

Ans. $\lim_{x\to 0} \frac{a^x-1+1-b^x}{x}$

$\lim_{x\to 0} \frac{(a^x-1)-(b^x-1)}{x}$

$\lim_{x\to 0} \frac{a^x-1}{x} - \lim_{x\to 0} \frac{b^x-1}{x}$

$= \ln a - \ln b$

$= \ln \frac{a}{b}$

⇒ Continuity :- (continuous)

$$\boxed{L.H.L = R.H.L = V.O.L}$$

Q.① check the continuity $f(x) = \begin{cases} \dfrac{\sin x}{\sin 3x} & \text{if, } x \neq 0 \\ \dfrac{1}{3} & \text{if, } x = 0 \end{cases}$ at $x = 0$

Ans.

L.H.L

$\lim_{x \to c^-} f(x)$

$\lim_{x \to 0^-} \dfrac{\sin x}{\sin 3x}$

$\therefore (x = 0-h,\ h \to 0)$

$\lim_{h \to 0} \dfrac{\sin (0-h)}{\sin 3(0-h)}$

$\lim_{h \to 0} \dfrac{\sin (-h)}{\sin (-3h)}$

$\lim_{h \to 0} \dfrac{\cancel{-}\sin h}{\cancel{-}\sin 3h}$

$\lim_{h \to 0} \dfrac{\frac{\sin h}{h}}{\frac{\sin 3h}{h}}$

$\dfrac{\lim_{h \to 0} \frac{\sin h}{h}}{3 \lim_{3h \to 0} \frac{\sin 3h}{3h}} = \dfrac{1}{3}$

R.H.L

$\lim_{x \to c^+} f(x)$

$\lim_{x \to 0^+} \dfrac{\sin x}{\sin 3x}$

$\therefore (x = 0+h,\ h \to 0)$

$\lim_{h \to 0} \dfrac{\sin (0+h)}{\sin 3(0+h)}$

$\lim_{h \to 0} \dfrac{\sin h}{\sin 3h}$

$\lim_{h \to 0} \dfrac{\frac{\sin h}{h}}{\frac{\sin 3h}{h}}$

$\dfrac{\lim_{h \to 0} \frac{\sin h}{h}}{3 \lim_{3h \to 0} \frac{\sin 3h}{3h}} = \dfrac{1}{3}$

V.O.L

at $x = 0$

$f(x) = \dfrac{1}{3}$

$\therefore$ L.H.L = R.H.L = function exists

L.H.L = R.H.L = V.O.L = continuous.

② check the continuity $f(x) = \begin{cases} 2x+1 & \text{if, } x < 1 \\ 3x & \text{if, } x = 1 \\ 4x^2-2 & \text{if, } x > 1 \end{cases}$ at $x = 1$

Ans

L.H.L

$\lim_{x \to c^-} f(x)$

$\lim_{x \to 1^-} (2x+1) \quad \therefore (x = 1-h,\ h \to 0)$

$\lim_{h \to 0} 2(1-h)+1$

$\lim_{h \to 0} 3-2h = 3$

R.H.L

$\lim_{x \to c^+} f(x)$

$\lim_{x \to 1^+} (4x^2-2) \quad \therefore (x = 1+h,\ h \to 0)$

$\lim_{h \to 0} 4(1+h)^2-2$

$= 2$

V.O.L

at $x = 1$

$f(x) = 3x = 3 \cdot 1 = 3$

L.H.L ≠ R.H.L (function does not exists)

L.H.L ≠ R.H.L ≠ V.O.L

(dis continuous)

③ check the continuity $f(x) = \begin{cases} \dfrac{x^2-a^2}{x-a} & \text{if } x \neq a \\ a & \text{if } x = a \end{cases}$ at $x=a$

Ans.

L.H.L

$\lim_{x \to c^-} f(x)$

$\lim_{x \to a^-} \dfrac{x^2-a^2}{x-a}$

$\therefore (x = a-h,\ h \to 0)$

$\lim_{h \to 0} \dfrac{(a-h)^2-(a)^2}{(a-h)-a}$

$= 2(a)^{2-1} = 2a$

R.H.L

$\lim_{x \to c^+} f(x)$

$\lim_{x \to a^+} \dfrac{x^2-a^2}{x-a}$

$\therefore (x = a+h,\ h \to 0)$

$\lim_{h \to 0} \dfrac{(a+h)^2-a^2}{(a+h)-a}$

$= 2(a)^{2-1} = 2a$

V.O.L

$\therefore$ at $x=a$

$f(x) = a$

$\therefore$ L.H.L = R.H.L = (function exists)

L.H.L = R.H.L $\neq$ V.O.L = (Discontinious)

④ check the continuity $f(x) = \begin{cases} (1+2x)^{1/x} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases}$ at $x=0$

Ans.

L.H.L

$\lim_{x \to c^-} f(x)$

$\lim_{x \to 0^-} (1+2x)^{1/x}$

$\therefore (x = 0-h,\ h \to 0)$

$\lim_{h \to 0} (1+2(0-h))^{1/(0-h)}$

$\lim_{h \to 0} (1-2h)^{1/-h}$

$\lim_{h \to 0} \left\{(1+(-2h))^{1/2h}\right\}^2$

$= e^2$

R.H.L

$\lim_{x \to c^+} f(x)$

$\lim_{x \to 0^+} (1+2x)^{1/x}$

$\therefore (x = 0+h,\ h \to 0)$

$\lim_{h \to 0} (1+2(0+h))^{1/h}$

$\lim_{h \to 0} (1+2h)^{1/h}$

$\lim_{h \to 0} \left\{(1+2h)^{1/2h}\right\}^2$

$= e^2$

V.O.L

at $x=0$

$f(x) = e^2$

$\therefore$ L.H.L = R.H.L = function exist

L.H.L = R.H.L = V.O.L = continuous

⑤ check the continuity $f(x) = \sin \frac{\pi [x]}{2}$ at $x=0$

Ans.

L.H.L

$\lim_{x \to c^-} f(x)$

$\lim_{x \to 0^-} \sin \frac{\pi [x]}{2}$

$\therefore (x = 0-h, h \to 0)$

$\lim_{h \to 0} \sin \frac{\pi [0-h]}{2}$

$\lim_{h \to 0} \sin \frac{\pi \times (-1)}{2}$

$\lim_{h \to 0} \sin (-\frac{\pi}{2})$

$-\lim_{h \to 0} \sin \pi/2 = -1$

R.H.L

$\lim_{x \to c^+} f(x)$

$\lim_{x \to 0^+} \sin \frac{\pi [x]}{2}$

$\therefore (x = 0+h, h \to 0)$

$\lim_{h \to 0} \sin \frac{\pi [0+h]}{2}$

$\lim_{h \to 0} \sin \frac{\pi \times 0}{2}$

$\lim_{h \to 0} \sin \frac{0}{2}$

$\lim_{h \to 0} \sin 0 = 0$

V.O.L

at $x = 0$

$f(x) = \sin \frac{\pi [x]}{2}$

$= \sin \frac{\pi [0]}{2}$

$= \sin \frac{0}{2}$

$= \sin 0 = 0$

$\therefore$ L.H.L $\neq$ R.H.L (function does not exist)

L.H.L $\neq$ R.H.L = V.O.L (discontinuous)

⑥ If $f(x) = \begin{cases} ax^2+b & \text{if } x<1 \\ 1 & \text{if } x=1 \\ 2ax-b & \text{if } x>1 \end{cases}$

It is continuous $x=1$, find a and b.

Ans

L.H.L

$\lim_{x \to c^-} f(x)$

$\lim_{x \to 1^-} (ax^2+b)$

$\therefore (x = 1-h, h \to 0)$

$\lim_{h \to 0} \{a(1-h)^2+b\}$

$= a+b$

R.H.L

$\lim_{x \to c^+} f(x)$

$\lim_{x \to 1^+} 2ax-b$

$\therefore (x = 1+h, h \to 0)$

$\lim_{h \to 0} \{2a(1+h)-b\}$

$= 2a-b$

V.O.L

at $x=0$

$f(x) = 1$

since the function is continuous $\Rightarrow$ L.H.L = R.H.L = V.O.L

$\Rightarrow a+b = 2a-b = 1$

$\Rightarrow a+b=1$

$\quad 2a-b=1$

$\quad 3a = 2 \Rightarrow a = 2/3$

$\Rightarrow a+b=1$

$\Rightarrow \frac{2}{3}+b=1$

$\Rightarrow b = 1-\frac{2}{3} = \frac{1}{3}$

(3) check the continuity

$$f(x) = [3x+11] \quad at \quad x = -11/3$$

Ans.

L.H.L

$\lim_{x \to c^-} f(x)$

$\lim_{x \to -11/3^-} [3x+11]$

$\therefore \left(x = \left(\frac{-11}{3} - h\right), h \to 0\right)$

$\lim_{h \to 0} \left[3\left(\frac{-11}{3} - h\right) + 11\right]$

$\lim_{h \to 0} [-11 - 3h + 11]$

$\lim_{h \to 0} [0 - 3h] = -1$

R.H.L

$\lim_{x \to c^+} f(x)$

$\lim_{x \to -11/3^+} [3x+11]$

$\therefore \left(x = \left(\frac{-11}{3} + h\right), h \to 0\right)$

$\lim_{h \to 0} \left[3\left(\frac{-11}{3} + h\right) + 11\right]$

$\lim_{h \to 0} [-11 + 3h + 11]$

$\lim_{h \to 0} [0 + 3h] = 0$

V.O.L

at $x = -11/3$

$f(x) = [3x+11]$

$= \left[\left(3 \times \frac{-11}{3}\right) + 11\right]$

$= [-11 + 11]$

$= 0$

L.H.L $\neq$ R.H.L (function does not exist)

L.H.L $\neq$ R.H.L = V.O.L (Discontinuous)

Notes-1

Topic: DIFFERENTIAL CALCULUS

DESCRIPTION : We define the slope of the curve y=f(x) at the point ,where x=a to be $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$,when it exists this limit is called the derivative of f at at x=a .now we will look at the derivative as a function derived from f by considering the limit(slope)at each point of the domain of f. The derivative of the function "f" with respect to the variable x is the function "f" whose value of x is f(x)=$\lim_{\to 0}\frac{f(x+h)-f(x)}{h}$.

A Derivative refers to the instantaneous rate of change of a quantity with respect to the others. That is denoted by dy/dx , Hear y=f(x).

Consider the general equation f=f(x).Let P&Q be two points of the graph whose abscissas are x and x+h. The corresponding ordinates are f(x)and f(x+h).The quantity h ,pictured in below as positive may be either positive or negative. In either case the slope of the secant line P&Q is 5=$\frac{f(x+h)-f(x)}{x+h-x}$

=$\frac{f(x+h)-f(x)}{h}$

Suppose now we keep "p" fixed and let "Q "move along the curve toward "p" (or let h approach zero).As this happens, the curve may be of such nature that the slope of the secant line various & approach some fixed value. In that case, the line through p with slope equal to this limiting value is called the target to the curve at p .Further, the lope of the tangent is said to be the slope of the curve. That is, the slope of the tangent, and also the slope of the curve, at the point p (x, y) is defined as $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ provided the limit exit.

MODEL QUESTIONS :

(1) Find the derivative of the X at x=1.

Ans: let f(x)=X then f'(1)=$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

=$\lim_{h\to 0}\frac{(1+h)-1}{h}$

=$\lim_{h\to 0}\frac{h}{h}$

=$\lim_{h\to 0} 1$

=1

Thus the derivative at x at x=1 is 1.

(2) Find the derivative of x^2 at x=1

Ans: let f(x)=x^2 then

f'(1)=$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

=$\lim_{h\to 0}\frac{[(1+h)^2]-([1]^2)}{h}$

=$\lim_{h\to 0}\frac{(1+2.1.h+h^2-1)}{h}$

= $\lim_{h\to 0}\frac{h(2+h)}{h}$

=$\lim_{h\to 0} 2+h=2$

(3) Derivative of constant function f(x)=c

Ans : $f'(c)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

=$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$

=$\lim_{h\to 0}\frac{c-c}{h}=0$

MOST PROBABLE QUESTONS:

(1) **F**ind the derivative of the x^3 at x=1.

(2) Find the derivative of x^n at x=1

(1) Find the derivative at 99x at x=100.

(2) Find the derivative of x^2-27

(3) Find the derivative of $\frac{1}{x^2}$

(4) Find the derivative of $2x^2-2$ at x=1

Notes-2

Topic : ALGEBRA OF DERIVATIVE

DESCRIPTION :

Now we define the algebra of derivative that is called the laws of derivative .consider two function f(x) and g(x) whose derivative in the same domain.
Here we define the algebraic operations of functions like addition ,substraction, multiplication ,scalar multiplication and division

Consider two function f(x) and g(x) in the same domain .then their operations

(1) Addition:$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
(2) Substraction :$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$
(3) Scalar multiplication:$\frac{d}{dx}[cf(x)] = c|f(x)|$
(4) Quotient of two function :$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f(x).\frac{d}{dx}[g(x)]-g(x).\frac{d}{dx}[f(x)]}{[g(x)]^2}$

MODEL QUESTIONS :

(1) Find the derivative of function $f(x) = 2x^2+3x+1$

Ans: $\frac{d}{dx}\big(f(x)\big) = \frac{d}{dx}(2x^2+3x+1)$

$= \frac{d}{dx}(2x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(1)$

$=4x+3+0$

=4x+3

MOST PROBABLE QUESTIONS:

(1) **F**ind the derivative of the following

(i) $8x^3$ (ii) $5x^2$

(2) find the derivative of the following functions.

(1) $5x^3+2x-3$

(2) 3xy

(3) $\frac{1}{x}$

(4) find the derivative of the function $\frac{3xy}{x}$

Notes -3

Topic: DERIVATIVE OF STANDARD FUNCTION(Trigonometric function)

DESCRIPTION :

Every one has already knows what is trigonometric function in 1st semester .it should be kept mind that to find the derivative of trigonometric function ,the angles must be in the radian measure . In case the given angle is measured in degrees, we must first convert it into radian measure by using the formula 180 degree =π radian.

We shall now find the derivative of trigonometric function using the definition of the derivative of the function.

(i) Derivative of sin x:

Let f(x)= sin x

Then f(x+h) = sin (x+h)

Thus f(x+h)-f(x)=sin(x+h)-sin x

$=\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-sinx}{h} = \frac{2\cos\left(\frac{2x+h}{2}\right)sinh/2}{h} = \cos\left(x+\frac{h}{2}\right)\frac{sinh/2}{h/2}$

Now taking limit h→ 0

$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}[\cos\left(x+\frac{h}{2}\right)\frac{\sin h/2}{h/2}$

$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right).\lim_{h\to 0}\frac{\sin h/2}{h/2}$

$= f'(x) = \cos x$ (where $\lim_{h\to 0}\frac{\sin h/2}{h/2} = 1$)

So we get that $\frac{d}{dx}(\sin x) = \cos x$

(ii) Derivative of cos x

Let f(x)=cos x Then f(x+h)=cos(x+h)-cos x

Since f(x+h)-f(x)=cos(x+h)-cos x

Or $\frac{f(x+h)-f(x)}{h} = \frac{\cos(x+h)-\cos x}{h} = \frac{-2\sin\left(x+\frac{h}{2}\right)sinh/2}{h}$

Since $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}[-\sin\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}]$

$=-\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right).\lim_{h\to 0}\frac{\sin h/2}{h/2}$

Then $f'(\cos x) = -sinx$

Similarly other functions are define.

(iii) $\frac{d}{dx}(\tan x) = sec^2 x$

(iv) $\frac{d}{dx}(cosec\ x) = -cosec\ x.\cot x$

(v) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(vi) $\frac{d}{dx}(\cot x) = -cosec^2 x$

MOST PROBABLE QUESTIONS:

(1) find the derivative of following function

(i) Y=$x^2 \tan x$

Ans:$\frac{d}{dx}(y) = \frac{d}{dx}(x^2 \tan x) = x^2.\frac{d}{dx}(\tan x) + \tan x.\frac{d}{dx}(x^2) = x^2.sec^2 x + \tan x.2x$

(ii) Y=$\sqrt{1+\sin 2x}$

Ans: y=$\sqrt{1+sin2x}$ =$\sqrt{(\cos x+\sin x)^2}$=$\cos x + \sin x$

$$\frac{\boldsymbol{dy}}{\boldsymbol{dx}} = \frac{d}{dx}(\cos x + \sin x) = \frac{d}{dx}(\cos x) + \frac{d}{dx}(\sin x) = -\sin x + \cos x$$

(2) find the derivative of the following functions

(i) Cot x, sec x, cosec x

(ii) X sin x

(iii) 5 tan x+ b cot x

(iv) X cos x+ sin x

(3) find the derivative of each of the following.

(i) $\sqrt{\cos x}$

(ii) $\frac{1-\tan x}{1+\tan x}$

(iii) $\frac{\tan x - \cos x}{\sin x.cos\, x}$

(iv) sec^2

(i) $\boldsymbol{x}^{2}\frac{1}{x\log_2 e}$+$\log_2 x.2x$+$\sec x.\tan x$

(ii) $\frac{-2cos\ x}{(1+sin)^2}$

Notes -4

Topic : DERIVATIVE OF EXPONENTIAL FUNCTIONS

DESCRIPTION :

The derivative of exponential function are denoted by e^x or a^x .

The exponential functions are important point in the derivation form. Which is denoted by

$\frac{d}{dx}(a^x) = a^x \log_e a$ and $\frac{d}{dx}(e^x) = e^x$.

Here we define how to calculate the derivative of exponential function .

(i) Derivative of a^x

Let $f(x) = a^x$, then $f(x+h) = a^{x+h}$

Thus $f(x+h) - f(x) = a^{x+h} - a^x = a^x(a^h - 1)$

Now $\frac{f(x+h)-f(x)}{h}$=$\frac{a^x(a^h-1)}{h}$

Proceeding the limit as h ends to 0,we have

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = a^x . \lim_{h\to 0} \frac{a^h - 1}{h}$$

$$\therefore [\lim_{h\to 0} \left(\frac{a^h - 1}{h}\right) = \log_e a]$$

$$\therefore f'(x) = a^x \log_e a$$

Similarly other is define.

MODEL QUESTIONS :

Find the derivative of the following function with respect to x.

(i) $\frac{d}{dx}(x^3 + e^x + \cot x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(e^x) + \frac{d}{dx}(\cot x) = 3x^2 + e^x - cosec^2 x$

(ii) $\frac{d}{dx}(\log_e x^3) = \frac{d}{dx}(3 \log_e x) = 3.\frac{d}{dx}(\log_e x) = \frac{3}{x}$

MOST PROBABLE QUESTIONS:

(1) find the derivative of the following function

(2) $\frac{\mathbf{3}}{\sqrt[3]{x}} - \frac{\mathbf{5}}{\boldsymbol{cos\ x}} + \log_e x + \frac{6}{sin\ x}$

(3) $\mathbf{3a^x}$

(1) find the derivative of each of the function.

(i) $x^2 - 7$

(ii) $\frac{1}{2\sqrt{\cos x}}(-\sin x)$

(iii) $a^x.2x + \sec x.\tan x$

(iv) $\frac{a^x(x\ ln\ a-1)+b^x(1-x\ln b)}{x^2}$

Notes -5

Topic: DERIVATIVE OF LOGARITHMIC FUNCTIONS

DESCRIPTION:

As the logarithmic function with base a(a>0,a$\neq$ 1) and exponential function with the same base form a pair of mutually inverse functions, the derivative of the logarithmic function can also be found be using the inverse function theorem.
First we should know the derivatives for the basic logarithmic functions.

(i) $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ (iii) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_a x}$

(ii) $\frac{d}{dx}\log_b x = \frac{1}{\ln(b).x}$

Let $f(x) = \log_a x \therefore f(x+h) = \log_a(x+h)$

$$\therefore f(x+h) - f(x) = \log_a(x+h) - \log_a x$$

$=\log_a\left(\frac{x+h}{h}\right) = \log_a(1+\frac{h}{x})$

$=\therefore \frac{f(x+h)-f(x)}{h} = \frac{1}{h}\log_a(1+\frac{h}{x})$

$=\frac{1}{x}.\frac{x}{h}\log_a\left(1+\frac{h}{x}\right) = \frac{1}{x}.\log_a(1+\frac{h}{2})^{\frac{x}{h}}$

$= \therefore \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{1}{x}.\lim_{h\to 0}(1+\frac{h}{x})^{\frac{x}{h}}$

$=\frac{1}{x}.\log_a \lim_{h\to 0}(1+\frac{h}{x})^{\frac{x}{h}}$

$\therefore f'(x) = \frac{1}{x}\log_a e$ [since $\lim_{h\to 0}(1+\frac{h}{x})^{\frac{x}{h}} = e$]

Hence we can written as $\log_a e = \frac{1}{\log_e a}$ $[since\ \log_a e . \log_e a = 1]$

Thus $\frac{d}{dx}(\log_a x) = \frac{1}{x\log a}$

<u>MODEL QUESTIONS :</u> find the derivative of the following problems

(1) Y=2 ln (3x^2-1)
Ans: let we put u=3x^2-1 then derivative of u is given by
U'=$\frac{du}{dx} = 6x$ so the final answer is $\frac{dy}{dx} = 2\frac{u'}{u} = 2\times\frac{6x}{3x^2-1} = \frac{12x}{3x^2-1}$

(2) Y=x($ln^3\ x$)

Ans: The notation y=$x(ln^3\ x)$ means y=$x\ (\ln x)^3$

This is the product of $x\ and\ (\ln x)^3$. so $\frac{dy}{dx} = x\frac{3(\ln x)^2}{x} + (\ln x)^3(1)$

$$=3(\ln x)^2 + (\ln x)^3$$

$$=(\ln x)^2(3 + \ln x)$$

MOST PROBABLE QUESTIONS:

find the derivative of the following functions

(1) $3\ln xy + \sin y = x^2$
(2) $y = (\sin x)^2$ by first taking logarithmic of each side of the equation .
(3) $y = \ln\,(cos\ x^2)$

find the derivative of the functions

(i) $y = \log_2 6x$
(ii) $y = 3\log_7(x^2 + 1)$
(1) Y=ln tan $\frac{x}{y}$
(2) Y=ln $(x + \sqrt{x^2 + a^2})$
(3) Y = ln $(\frac{1}{\sqrt{1-x^4}})$

Notes -6

Topic: DERIVAT IVE OF SOME STANDARD FUNCTIONS

DESCRIPTION:

We can algebraically find the derivative of all standard function. That is included exponential function ,trigonometric function, exponential function, logarithmic function and inverse trigonometric function.

Previously we discuss most of all types of derivative functions. Here we discuss the inverse trigonometry function. Lets discuss some standard formulas .

(1) $\frac{d}{dx}(x^n) = nx^{n-1}$
(2) $\frac{d}{dx}(a^x) = a^x \log_e a$
(3) $\frac{d}{dx}(\log_a x) =$
(4) $\frac{d}{dx}(e^x) =$
(5) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
(6) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
(7) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
(8) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
(9) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{\sqrt{x^2(\sqrt{x^2-1})}}$
(10) $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{\sqrt{x^2(\sqrt{x^2-1})}}$

MODEL QUESTIONS: Find the derivative of following functions.

(1) Y=$\sin^{-1} 2x$

Ans: $\frac{d}{dx}(\sin^{-1} 2x) = \frac{1}{\sqrt{1-(2x)^2}} = \frac{1}{\sqrt{1-4x^2}}$

(2) Y=$\sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$ find $\frac{dy}{dx}$

Ans: putting x=$\sin\theta$ and $\sqrt{x} = \sin\varphi$

We get y=$\sin^{-1}[sin\theta \cos\varphi - \sin\varphi \cos\theta$

= $\sin^{-1}[\sin(\theta - \varphi)] = (\theta - \varphi) = \sin^{-1} x - \sin^{-1}\sqrt{x}$

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x - \sin^{-1}\sqrt{x}) = \frac{d}{dx}(\sin^{-1} x) - \frac{d}{dx}(\sin^{-1}\sqrt{x})$

=$[\frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}.\sqrt{1-x}}]$

MOST PROBABLE QUESTIONS:

find the derivative of the followings:

(1) Y=$\tan^{-1}\sqrt{x}$
(2) Y=$\cos^{-1}(\cot x)$
(3) Y=$\cos^{-1}(\tan x)$

differentiate the following functions:

(1) Y=$\sqrt{\cot^{-1}\sqrt{x}}$
(2) If Y=$\frac{x\sin^{-1}x}{\sqrt{1-x^2}}$ $find\ \frac{dy}{dx}$
(3) $y = (\frac{1-\cos x}{\sin x})$

Find the derivative of the following functions

(1) $\tan^{-1}(\sqrt{\frac{1-\cos x}{1+\cos x}})$
(2) $\tan^{-1}(\sec x + \tan x)$
(3) $\cos^{-1}(\sqrt{\frac{1+\cos x}{2}}$

Notes -7

Topic: DERIVATIVE OF COMPOSITE FUNCTION(CHAIN RULE)

DESCRIPTION:

We have been differentiating y, a function of x with respect to x. We also comes across since u when y is a function of 'u' and 'u' is a function of x that is y=f(u) and u=g(x) then y=f|g(x)| in this case we say y is a function of a function or y is a composite function. We shall now find in method of differentiating composite function. (chain rule) if u is a function of y define by y=f(u) and u is a function of x define by u=g(x), then y is a function of x and $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

Lets more describe about chain rule. Suppose that we have two functions f(x) and g(x) and they are both differentiable .

(1) If we define F(x) =(f o g)(x) then the derivative of F(x) is F'(x)= f'(g(x) g'(x))
(2) If we have y=f(u) and u =g(x) then derivative of y is $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$
Corollary :if y=f(u) , u=g(v) and v=h(x) then $\frac{dy}{dx} = \frac{dy}{du}.\frac{dy}{dv}.\frac{dv}{dx}$.
By using these formulas we solve the problems.

MODEL QUESTIONS : solve these problems by using the chain rule.

(1) Y=$(2x^3 - 1)^4$ $find\ \frac{dy}{dx}$
Ans: let u=$2x^3 - 1$ $\therefore y = u^4$ then $\frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = 6x^2$
By chain rule ,$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 4u^3.6x^2 = 24x^2(2x^3 - 1)^3$.
(2) Y=$\sqrt{ax^2 + bx + c}$
Ans: let u=a$x^2 + bx + c$

then y=$\sqrt{u}$ $then$ $\frac{dy}{du} = \frac{d}{du}(\sqrt{u}) = \frac{1}{2\sqrt{u}}$

and $\frac{du}{dx} = \frac{d}{dx}(ax^2 + bx + c) = a.2x + b.1 + 0$

=2ax+b

By chain rule $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{2\sqrt{u}}.(2ax + b) = \frac{2ax+b}{2\sqrt{ax^2+bx+c}}$

<u>MOST PROBABILITY QUESTIONS:</u> find $\frac{dy}{dx}$

(1) $y = \sqrt{\frac{1-\tan x}{1+\tan x}}$

(2) $y = \frac{1}{(x^3+\sin x)^2}$

(3) $y = \ln(\sqrt{x} + 1)$

(1) $y = (\frac{7x}{x^2+1})^3 \; find \; \frac{dy}{dx}$

(2) $if \; f(x) = sin^3 x, find \; f'(x)$

(3) $find \; the \; derivative \; of \; \sin x^0$

using chain rule to find the derivative of the following.

(1) $e^{sin^2} x$

(2) $\sqrt{e^{\sqrt{x}}}$

(3) Log(log x)

Notes -8

TOPIC: DERIVATIVE OFCOMPOSITE FUNCTION(CHAIN RULE)

DESCRIPTION:

Already we discuss chain rule in previous lecture now we discus some extra problem., We have been differentiating y, a function of x with respect to x. We also comes across since u when y is a function of 'u' and 'u' is a function of x that is y=f(u) and u=g(x) then y=f|g(x)| in this case we say y is a function of a function or y is a composite function. We shall now find in method of differentiating composite function.
(chain rule) if u is a function of y define by y=f(u) and u is a function of x define by u=g(x), then y is a function of x and $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

Lets more describe about chain rule. Suppose that we have two functions f(x) and g(x) and they are both differentiable .

(1) If we define F(x) =(f o g)(x) then the derivative of F(x) is F'(x)= f'(g(x) g'(x))
(2) If we have y=f(u) and u =g(x) then derivative of y is $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$
Corollary :if y=f(u) , u=g(v) and v=h(x) then $\frac{dy}{dx} = \frac{dy}{du}.\frac{dy}{dv}.\frac{dv}{dx}$.
By using these formulas we solve the problems .

MODEL QUESTIONS :

(1) Differentiate $sin^2\ x^3\ by\ using\ chain\ rule.$
Ans: Let y=$n^2$and u=sin x^3
That is y=n^2,u=sin v and v=x^3
Applying these chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$$

Now $\frac{dy}{du} = 2u$,$\frac{du}{dv} = \cos v$ and $\frac{dv}{dx} = 3x^2$
By putting these value to the above equation we get that $\frac{dy}{dx} = 2u.\cos v.3x^2 = 6x^2 \sin x^3 \cos x^3$.

MOST PROBABLE QUESTIONS:

using chain rule to find the derivative of each of the following

(1) $[\tan(3x^2+5)]^8$
(2) $\sqrt{\tan x}$
(3) $(\frac{2\tan x}{\tan x+\cos x})^2$
(1) $\log(\frac{1-x}{1+x})$
(2) $\log(x+\sqrt{x^2+a})$
(3) $\frac{\log x}{1+x\log x}$

find the derivative of the following functions by using chain rule.

(1) $(3x^2+2x+1)^8$
(2) $(x^2+3)^4$
(3) Sin 6x + cos 7x

Notes -9

TOPIC : DERIVATIVE OF COMPOSITE FUNCTION (CHAIN RULE)

DESCRIPTION:

Already we discuss chain rule in previous lecture now we discus some extra problem., We have been differentiating y, a function of x with respect to x. We also comes across since u when y is a function of 'u' and 'u' is a function of x that is y=f(u) and u=g(x) then y=f|g(x)| in this case we say y is a function of a function or y is a composite function

We shall now find in method of differentiating composite function.

(chain rule) if u is a function of y define by y=f(u) and u is a function of x define by u=g(x) then y is a function of x and $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

Lets more describe about chain rule. Suppose that we have two functions f(x) and g(x) and they are both differentiable .

(1) If we define F(x) =(f o g)(x) then the derivative of F(x) is F'(x)= f'(g(x) g'(x))
(2) If we have y=f(u) and u =g(x) then derivative of y is $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$
Corollary :if y=f(u) , u=g(v) and v=h(x) then $\frac{dy}{dx} = \frac{dy}{du}.\frac{dy}{dv}.\frac{dv}{dx}$.
By using these formulas we solve the problems .

MODEL QUESTIONS : differentiate each of the following with respect to x.

(1) $logx.e^{\sin x+x^3}$
Ans: let $y = \log x.e^{\sin x+x^3}$

By using product rule ,

$\frac{dy}{dx} = \log x.\frac{d}{dx}(e^{\sin x+x^3})+e^{\sin x+x^3}.\frac{d}{dx}(\log x)$

= $\log x.e^{\sin x+x^3}.(\cos x + 3x^2) + e^{\sin x+x^3}.\frac{1}{x}$

= $e^{\sin x+x^3}[(\cos x + 3x^2)\log x + \frac{1}{x}]$

(2) $\log[\log(\log x)]$
Ans: let $y = \log[\log(\log x)]$
Then $\frac{dy}{dx} = \frac{1}{\log(log\ x)}.\frac{d}{dx}\log(\log x)$
= $\frac{1}{\log(\log x)}.\frac{1}{\log x}.\frac{d}{dx}(\log x)$
= $\frac{1}{\log(\log x).(\log x)}.\frac{1}{x}$

(3) find the differential coefficient of $\sin[\cos(tanx)]$

Ans: let $y = \sin[\cos(\tan x)]$

$$\frac{dy}{dx} = \cos[\cos(\tan x)].\frac{d}{dx}[\cos(tanx)]$$

= $\cos[\cos(\tan x)][-\sin(\tan x)].\frac{d}{dx}(\tan x)$

$= -\cos[\cos(\tan x)[\sin(\tan x)].sec^2 x$

MOST PROBABLE QUESTIONS:

using chain rule to find the derivative of each of the following.

(1) $\log[(sinx)^{\cos x}$

(2) $\sqrt{\cot x}$

(3) $(3x^4 - 2$

(4) $\sqrt{\sin x}$

(5) $\log(\sin x)$

(6) $cos^2\sqrt{x}$

find the differential coefficient of

(1) $x^3 sin^4 x(\log x)^5$

(2) $\log\{x - 3\sqrt{x^2 - 6x + 1}\}$

Notes -10

TOPIC : DIFFERENTIATION

DESCRIPTION: Let $f(x)$ be a real function and a be any number .Then we define

(i) Right-Hand Derivative:

$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ if it exits is called the right-hand derivative of f(x) at x=a and it Is denoted by Rf'(a).

(ii) Left-Hand Derivative:

$\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$ if it exits, is called the left-hand derivative of f(x) at x=a and it is denoted by Lf'(a).

(Differentiability)

A function f(x) is said to be a differentiable at x=a, if Rf'(a)=Lf'(a). If, however Rf'(a)≠Lf'(a),we say that f(x) is not differentiable at x=a.

(Relation between continuity and Differentiability)

Every differentiable function is continuous ,but every continuous function is not differentiable .

Proof: let f(x) be a differentiable function and let a be any real number in its domain.
Then,$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a)$

Now , $\lim_{h\to 0}[f(a+h) - f(a)$

= $\lim_{h\to 0}[\frac{f(a+h)-f(a)}{h} \times h]$

=$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \times \lim_{h\to 0} h$

=$f'(a) \times 0 = 0$

Thus $\lim_{h\to 0}[f(a+h) - f(a)]$=0

$\lim_{h\to 0} f(a+h) = f(a)$.

MODEL QUESTIONS : show that the function $f(x) = x^2$ is differentiable at x=1

Ans: $Rf'(x)$=$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{(1+h)^2-(1)^2}{h}$

$=\lim_{h\to 0} \frac{1+h^2+2h-1}{h} = \lim_{h\to 0}(h+2) = 2$

$= \lim_{h\to 0} \frac{1+h^2-2h-1}{-h} = \lim_{h\to 0}(-h+2) = 2$

$\therefore Rf'(1) = Lf'(1) = 2$

this show that f(x) is differentiable at x=1 and f'(1)=2

MOST PROBABLE QUESTIONS:

(1) Show that f(x)=$[x]$ is not differentiable at x=1
(2) Show that the constant function is differentiable or not .
(3) Show that f(x)=x is differentiable or not at x=1

(4) Show that the function $f(x) = \{1+x\,, if\ x \leq 2\ |5-x, if\ x > 2\}$ is not differentiable at x=1.

(5) Show that $f(x) = |x|$ is differentiable or not at x=0.

(6) Show that $f(x) = \log x$ is differentiable or not at x=0.

(7) Show that the function $f(x) = \left\{x \sin\frac{1}{x}, when\ x \neq 0 \middle| 0, when\ x = 0\right\}$ is continuous but not differentiable at x=0.

(8) Show that the function $f(x) = \left\{x^2 \cos\frac{1}{x}, when\ x \neq 0 \middle| 0, when\ x = 0\right\}$ is weather continuous or differentiable or both at x=0.

Notes-11

TOPIC : METHODE OF DIFFERENTIATION (parametric function)

DESCRIPTION:

Some times both x and y may be given as functions of another variable called a parameter.

For example ,any point (x, y) on the circle $x^2 + y^2 = r^2$ can be given by $x = rcos\ t, y = r\sin t$, the variable quantity t is called parameter. The function consider these variable is called parametric function.

The term parameter is also used to mean a quantity which is invariable for a given curve but changes when we move from the curve of a given type to another. In such case the derivative is given in terms of the variable parameter .In such case the derivative is given in Sterms of the variable parameter

We shall n0w discuss the method of finding $\frac{dy}{dx}$ $when\ x\ and\ y\ are\ function\ of\ t.$

Let $x = f(t) and\ y = g(t)$ corresponding to an increment δt in t, there are increments $\delta x\ and\ \delta y$ in x and y respectively.

Then $x + \delta x = f(t + \delta t)\ and\ \ y + \delta y = g(t + \delta t)$

$$\therefore\ \delta x = f(t + \delta t) - f(t)$$

And $\delta y = g(t + \delta t) - g(t)$ from these two equation combine we get

$$\frac{\delta y}{\delta x} = \frac{g(t + \delta t) - g(t)}{f(t + \delta t) - f(t)} = \frac{g(t + \delta t) - g(t)}{f(t + \delta t) - f(t)} \cdot \frac{\delta t}{\delta t}$$

$= \frac{g(t+\delta t)-g(t)}{\delta t} \cdot \frac{\delta t}{f(t+\delta t)-f(t)}$

Now as $\delta t \to 0, \delta x \to 0\ and\ \delta y \to 0$

$$\therefore\ \lim_{\delta t\to 0} \frac{g(t + \delta t) - g(t)}{\delta t} \div \lim_{\delta t\to 0} \frac{f(t + \delta t) - f(t)}{\delta t}$$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{dy/dt}{dx/dt}$

MODEL QUESTIONS :

(1) If $x = at^2$ and $y = 2bt$, find $\frac{dy}{dx}$
Ans: Here $x = at^2$ and $y = 2bt$

$\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2b$ then $\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx} = \frac{2b}{2at} = \frac{b}{at}$.

(2) If $y = a\cos\theta$ and $x = a(\theta + \sin\theta)$,find $\frac{dy}{dx}$

Ans: $y = a\cos\theta \; and \; x = a(\theta + \sin\theta)$

$\frac{dy}{d\theta} = -a\sin\theta$, and $\frac{dx}{d\theta} = a(1 + \cos\theta)$

$$\frac{dy}{dx} = \frac{dy}{d\theta} . \frac{d\theta}{dy}$$

= $\frac{-\text{a}\sin\theta}{a(1+\cos\theta)}$

= $\frac{-2\sin\frac{\theta}{2} . \; cos\frac{\theta}{2}}{2\; cos^2\frac{\theta}{2}} = -\tan\frac{\theta}{2}$

MOST PROBABLE QUESTIONS:

(1) Find the derivative of $x = a\; sin^3 t \; and \; y = b\; cos^3 t$

(2) $x = \frac{2at}{1+t^2} \; and \; y = \frac{2bt}{1-t^2}$ find the derivative.

(3) Find the derivative of the function $x = \frac{1-t^2}{1+t^2} \; and \; y = \frac{2t}{1+t^2}$

(4) $\boldsymbol{x} = at^2 \; and \; y = at^3$

(5) $x = \frac{a(1-t)}{1+t^2}$ and $y = at(\frac{1-t^2}{1+t^2})$

(6) $x = a\left(\theta + \frac{1}{\theta}\right) and \; y = a(\theta - \frac{1}{\theta})$

Notes -12

TOPIC :DIFFERENTIATION OF PARAMETRIC FUNCTION

DESCRIPTION:

Some times both x and y may be given as functions of another variable called a parameter.

For example ,any point (x, y) on the circle $x^2 + y^2 = r^2$ can be given by $x = rcos\ t, y = r \sin t$, the variable quantity t is called parameter. The function consider these variable is called parametric function.

The term parameter is also used to mean a quantity which is invariable for a given curve but changes when we move from the curve of a given type to another. In such case the derivative is given in terms of the variable parameter .In such case the derivative is given in terms of the variable parameter .

We shall n0w discuss the method of finding $\frac{dy}{dx}$ $when\ x\ and\ y\ are\ function\ of\ t.$

Let $x = f(t) and\ y = g(t)$ corresponding to an increment δt in t, there are increments $\delta x\ and\ \delta y$ in x and y respectively.

Then $x + \delta x = f(t + \delta t)\ and\ \ y + \delta y = g(t + \delta t)$

$$\therefore\ \delta x = f(t + \delta t) - f(t)$$

And $\delta y = g(t + \delta t) - g(t)$ from these two equation combine we get

$$\frac{\delta y}{\delta x} = \frac{g(t + \delta t) - g(t)}{f(t + \delta t) - f(t)} = \frac{g(t + \delta t) - g(t)}{f(t + \delta t) - f(t)} \cdot \frac{\delta t}{\delta t}$$

$= \frac{g(t+\delta t)-g(t)}{\delta t} \cdot \frac{\delta t}{f(t+\delta t)-f(t)}$

Now as $\delta t \to 0, \delta x \to 0\ and\ \delta y \to 0$

$$\therefore\ \lim_{\delta t \to 0} \frac{g(t + \delta t) - g(t)}{\delta t} \div \lim_{\delta t \to 0} \frac{f(t + \delta t) - f(t)}{\delta t}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{{dy}/{dt}}{dx/dt}$$

MODEL QUESTIONS :

(1) find $\frac{\mathrm{dy}}{\mathrm{dx}}, x = \theta + \sin\theta, y = 1 + \cos\theta\ at\ \theta = \frac{\pi}{4}$

Ans: Here $x = \theta + \sin\theta$ then $\frac{dx}{d\theta} = 1 + \cos\theta$

Again $y = 1 + \cos\theta$ $then$ $\frac{dy}{d\theta} = -\sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = \frac{-\sin\theta}{1+\cos\theta}$$

$\left[\frac{dy}{dx}\right]$ $at\ \theta = \frac{\pi}{4}$ = $\frac{-\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}$ = $\frac{-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{-1}{1+\sqrt{2}}$.

(2) Find $\frac{dy}{dx}$, if $x = 3\cot t - 2\ cos^3 t$, $y = 3\sin t - 2sin^3 t$

Ans: Given $x = 3\cot t - 2cos^3 t$

Then $\frac{dx}{dt} = -3\sin t - 3(cos^2 t).(-\sin t)$

= $-3\sin t + 6\ cos^2 t.\sin t = 3\sin t.\cos 2t$

Again, $y = 3\sin t - 2\ sin^3 t$

Then $\frac{dy}{dt} = 3\cos t - 6\ sin^2 t\ .\cos t$

=$3\cos t(1 - 2\ sin^2 t) = 3\cos t.\cos 2t$

$\frac{\mathrm{dy}}{\mathrm{dx}}$= $\frac{dy}{dt}.\frac{dt}{dx}$ = $\frac{3\cos t.\cos 2t}{3\sin t.\cos 2t}$

MOST PROBABILITY QUESTIONS:

(1) Find $\frac{dy}{dx}$, if $sinx = \frac{2t}{1+t^2}$, $\tan y = \frac{2t}{1-t^2}$

(2) $find\ \frac{dy}{dx}\ where\ y = x^4 \log x$

(3) What is the derivative of $x|x|at\ x = 2$

(4) $x = a(\theta + sin\theta)\ and\ y = a(1 - \cos\theta)$

(5) $x = \frac{at^2}{1+t^2}\ and\ y = \frac{at^3}{1+t^2}$

(6) $x = a\sqrt{\frac{t^2-1}{t^2+1}}\ and\ y = at\sqrt{\frac{t^2-1}{t^2+1}}$

Notes -13

TOPIC :DIFFERENTIATION OF FUNCTION WITH RESPECT TO FUNCTION

DESCRIPTION: IF $Y = f(X)$ is differentiable , then the derivative of y with respect to x is

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

If f and g are differentiable functions of x and if $\frac{df}{dg} = \frac{\frac{df}{dx}}{\frac{dg}{dx}} = \frac{f'(x)}{g'(x)}$

The understanding of the differentiation of the function with respect to a function
IF $Y = f(X)$ is differentiable , then the derivative of y with respect to x is

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

If f and g are differentiable functions of x and if $\frac{df}{dg} = \frac{\frac{df}{dx}}{\frac{dg}{dx}} = \frac{f'(x)}{g'(x)}$

Which is same as the definition.

Suppose we have two differentiable functions given by $y = f(x)$ and $z = g(x)$. To find the derivative of y with respect to z .we regard x as a parameter and find

$$\frac{dy}{dx} = f'(x), and\ \frac{dz}{dx} = g'(x)$$

i.e. $\frac{dy}{dz} = \frac{dy}{dx}.\frac{dx}{dz} = \frac{f'(x)}{g'(x)}$

MODEL QUESTIONS :

(1) Differentiate $\tan^{-1} x$ w.r.t $\tan^{-1}\sqrt{1+x^2}$.
Ans: Let y= $\tan^{-1} x$ and z=$\tan^{-1}\sqrt{1+x^2}$
$\therefore \frac{dy}{dx}=\frac{1}{1+x^2}, \frac{dz}{dx}=\frac{2x}{2(1+1+x^2)\sqrt{1+x^2}} = \frac{x}{(2x+x^2)\sqrt{1+x^2}}$
$\therefore \frac{dy}{dz}=\frac{dy}{dx}.\frac{dx}{dz}$
$= \frac{1}{(1+x^2)}.\frac{(2+x^2)\sqrt{1+x^2}}{x} = \frac{2+x^2}{x\sqrt{1+x^2}}$

(2) Differentiate $\sin^{-1}(\frac{2x}{1+x^2})$ w.r.t $\cos^{-1}(\frac{1-x^2}{1+x^2})$
Ans: Set x= $\tan\theta$ in both the expressions
Let y = $\sin^{-1}(\frac{2x}{1+x^2})$ and z = $\cos^{-1}(\frac{1-x^2}{1+x^2})$
Y = $\sin^{-1}(\frac{2tan\theta}{1+tan^2\theta})$ and z= $\cos^{-1}(\frac{1-tan^2\theta}{1+tan^2\theta})$
Y=$\sin^{-1}(sin2\theta)$ and z = $\cos^{-1}(cos2\theta)$
Y= 2θ and z=2θ
Y = $2\tan^{-1} x$ and z = $2\tan^{-1} x$
$\frac{dy}{dx} = \frac{2}{1+x^2}$ and $\frac{dz}{dx} = \frac{2}{1+x^2}$

$$\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = \frac{2}{(1+x^2)} \frac{(1+x^2)}{2} = 1.$$

MOST PROBABLE QUESTIONS:

(1) Differentiate $sin^2 x$ w.r.t. $(\ln x)^2$.

(2) Differentiate $e^{\tan x}$ w.r.t. sin x.

(3) Differentiate $e^{sin^{-1}}$ w.r.t. $e^{-cos^{-1} x}$

Notes-14

TOPIC:DIFFERENTIATION WITH RESPECT TO A FUNCTION

DESCRIPTION:IF $Y = f(X)$ is differentiable , then the derivative of y with respect to x is

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

If $f\ and\ g$ are differentiable functions of x and if $\frac{df}{dg} = \frac{\frac{df}{dx}}{\frac{dg}{dx}} = \frac{f'(x)}{g'(x)}$

The understanding of the differentiation of the function with respect to a function

IF $Y = f(X)$ is differentiable , then the derivative of y with respect to x is

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

If $f\ and\ g$ are differentiable functions of x and if $\frac{df}{dg} = \frac{\frac{df}{dx}}{\frac{dg}{dx}} = \frac{f'(x)}{g'(x)}$

Which is same as the definition.

Suppose we have two differentiable functions given by $y = f(x)$ and $z = g(x)$. To find the derivative of y with respect to z .we regard x as a parameter and find

$$\frac{dy}{dx} = f'(x), and\ \frac{dz}{dx} = g'(x)$$

i.e. $\frac{dy}{dz} = \frac{dy}{dx}.\frac{dx}{dz} = \frac{f'(x)}{g'(x)}$

MODEL QUESTION:

(1) Differentiate $\tan^{-1} x\ \ w.r.t\ \cos^{-1} x$

Ans : let $y = \tan^{-1} x$ and $z = \cos^{-1} x$

We have to find $\frac{dy}{dz}$. Now

$\frac{dy}{dx} = \frac{1}{1+x^2}$ and $\frac{dz}{dx} = \frac{-1}{\sqrt{1-x^2}}$

$$\therefore \frac{dy}{dz} = \frac{dy}{dx}.\frac{dx}{dz} = \frac{-\sqrt{1-x^2}}{1+x^2}$$

(2) Differentiate $e^{\sin x}\ w.r.t \cos x$

Ans: let $y = e^{\sin x}\ and\ z = \cos x$

We have to find $\frac{dy}{dz} = \frac{dy}{dx}.\frac{dx}{dz} = e^{\sin x}.\cos x \times \frac{-1}{\sin x} = -e^{\sin x}.\cot x$

(3) Differentiate $\frac{1-\cos x}{1+\cos x}\ w.r.t\ \frac{1-\sin x}{1+\sin x}$

Ans: let $y = \frac{1-\cos x}{1+\cos x}$ and $z = \frac{1-\sin x}{1+\sin x}$

Now, $\frac{dy}{dx} = \frac{\sin x(1+\cos x)+\sin x(1-\cos x)}{(1+\cos x)^2}$

= $\frac{\sin x+sinx \cos x+\sin x(1-\cos x)}{(1+\cos x)^2} = \frac{2\sin x}{(1+\cos x)^2}$

$$\frac{dz}{dx} = \frac{-\cos(1+sinx)-\cos x(1-\sin x)}{(1+\sin x)^2} = \frac{-2\cos x}{(1+\sin x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{dz}$$

$$\frac{2\sin x}{(1+\sin x)^2} \cdot \frac{(1+\sin x)^2}{(-2\cos x)} = -\tan x \frac{(1+\sin x)^2}{(1+\cos x)^2}$$

MOST PROBABLE QUESTIONS:

(1) Differentiate $\sqrt{x}$ with respect to x^2.

(2) Differentiate sin x w.r.t cos x.

(3) Differentiate $\sin^{-1}(\frac{2x}{1+x^2})$ w.r.t $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(4) Differentiate $(\frac{\tan^{-1} x}{1+\tan^{-1} x})$ w.r.t $\tan^{-1} x$.

(5) Differentiate $\tan^{-1}(\frac{2x}{1-x^2})$ w.r.t $\sin^{-1}(\frac{2^x}{1+x^2})$.

(6) Differentiate $\sin^{-1}(\frac{2x}{1+x^2})$ w.r.t $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$.

(7) Differentiate $\ln(\sin x)$ w.r.t tan x.

(8) Differentiate $e^{\sin^{-1} x}$ w.r.t $e^{-\cos^{-1} x}$.

(9) Differentiate $\frac{1-\cos x}{1+\cos x}$ w.r.t $\frac{1-\sin x}{1+\sin x}$.

Notes-15

TOPIC:DIFFERENTIATION OF IMPLICITY FUNCTION

DESCRIPTION:

In mathematics ,an implicit equation is a relation of the form R($x_{1,\ldots\ldots\ldots,}x_n$)=0, where R is a function of several variable (often a polynomial). For example , the implicit equation of the u it circle is $x^2 + y^2$-1=0.
Sometimes relationships cannot be represented by an explicit function . For example ,$x^2 + y^2 = 1$.Implicithe differentiation helps us find dy/dx even for relationships like that. This is done using the chain rule, and viewing y as an implicit function of x. For example, according to the chain rule, the derivative of y^2 would be 2y.(dy/dx).

An implicit function is a function that is defined implicit by an implicit equation by associating one of the variables (the value) with the others (the arguments) Thus ,an implicit function for y is the context of the unit circle is defined implicity by x^2+f$(x)^2$-1 = 0 . The implicit function is defines f as a function of x only if -1≤ x≤ 1 and one considers only non-negative (or non-positive) values of the function.

MODEL QUESTION:

(1) Find $\frac{dy}{dx}$, when $x^{2+}y^2$= 2axy
Ans: given equation is $x^2 + y^2$=2axy
Differentiating w.r.t. x
$\boldsymbol{\frac{d}{dx}(x^2) + \frac{d}{dx}(y)^2 = 2a\,\frac{d}{dx}(xy)}$
$=> \ 2x + 2y\ \frac{dy}{dx} = 2a\ [x.\frac{dy}{dx} + y]$
$=>x+y\frac{dy}{dx} = ax\frac{dy}{dx} + ay \ => y\frac{dy}{dx} - ax\frac{dy}{dx} = ay - x$
$=>(y-ax)\frac{dy}{dx} = ay - x \ => \frac{dy}{dx} = \frac{ay-x}{y-ax}.$

(2) Find $\frac{dy}{dx}, e^y\, lnx + lny = 0$
Ans: Given equation is $e^y lnx + lny = 0$
Differentiating w.r.t. x, we get

$$(e)^y\frac{d}{dx}(lnx) + lnx.\frac{d}{dx}(e^y) + x.\frac{d}{dx}(lny) + lny.\frac{d}{dx}(x) = 0$$

$$=> \ e^y.\frac{1}{x} + lnx.e^y.\frac{d}{dx} + x.\frac{1}{y}\frac{dy}{dx} + lny = 0$$

$$=> \frac{dy}{dx}\left[lnx.e^y + \frac{x}{y}\right] = -\left(\frac{e^y}{x} + lny\right)$$

$$=> \ \frac{dy}{dx} = -\frac{\left(\frac{e^y+xlny}{x}\right)}{\frac{ylnx.e^y+x}{y}} = -\frac{y(e^y + xlny)}{x(ylnxe^y + x)}$$

<u>MOST PROBABLE QUESTIOS:</u>

Find the derivative of y w.r.t x

(1) $ax^2 + by^2 = 25$
(2) $\frac{x^2}{9} + \frac{y^2}{16} = 1$
(3) $e^{xy} + \ ysinx = 1$

(4) $\boldsymbol{x^3 + y^3 = 3axy}$

(5) $x^y = \ e^{x-y}$

(6) y tan x-$y^2\ cosx + 2x = 0$

(7) tan(x+y)+ tan(x-1)=0

(8) $x^{\frac{1}{2}}\ y^{-\frac{1}{2}} + x^{\frac{3}{2}}\ y^{\frac{-3}{2}} = 0$

(9) $\ln\sqrt{x^2 + y^2 = \tan^{-1}(\frac{y}{x})}$

Notes-16

TOPIC: DIFFERENTIATION OF IMPLICITY FUNCTION

DESCRIPTION :

In mathematics ,an implicit equation is a relation of the form R($x_{1,.........,}x_n$)=0, where R is a function of several variable (often a polynomial). For example , the implicit equation of the u it circle is $x^2 + y^2$-1=0.

Sometimes relationships cannot be represented by an explicit function . For example ,$x^2 + y^2 =$ 1.Implicithe differentiation helps us find dy/dx even for relationships like that.This is done using the chain rule, and viewing y as an implicit function of x. For example,according to the chain rule, the derivative of y^2 would be 2y.(dy/dx).

An implicit function is a function that is defined implicit by an implicit equation by associating one of the variables (the value) with the others (the arguments) Thus ,an implicit function for y is the context of the unit circle is defined implicity by x^2+f$(x)^2$-1 = 0 . The implicit function is defines f as a function of x only if -1≤ x≤ 1 and one considers only non-negative (or non-positive) values of the function.

MODEL QUESTION:

(1)If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, find\ \frac{dy}{dx}$.

Ans: Given that

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots(1)$$

Differentiating both sides of (i) with respect to x , we get :

2ax+2h{x.$\frac{dy}{dx}.y.1\} + 2by.\frac{dy}{dx} + 2g + 2f.\frac{dy}{dx} = 0$

Or (2ax+2by+2g)+(2hx+2by+2f).$\frac{dy}{dx} = 0$

$$\therefore\ \frac{dy}{dx} = -\left(\frac{ax + hy + g}{hx + by + f}\right).$$

(1) If $\sqrt{1 - x^2} + \sqrt{1 - y^2}$=a(x-y), prove that $\frac{dy}{dx}$= $\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

Ans: Given that

$\sqrt{1 - x^2} + \sqrt{1 - y^2 = a(x - y)}$ ....(1)

Putting x = sin θ and y=sin $\emptyset, it\ becomes$

cos$\theta + cos\emptyset = a(sin\theta - sin\emptyset)$

or $\frac{cos\theta+cos\emptyset}{sin\theta-sin\emptyset} = a$

or $\frac{2\cos\left(\frac{\theta+\emptyset}{2}\right)\cos\left(\frac{\theta+\emptyset}{2}\right)}{2\cos\left(\frac{\theta+\emptyset}{2}\right)+\sin\left(\frac{\theta+\emptyset}{2}\right)}$ =a

$$\therefore \cot\left(\frac{\theta-\emptyset}{2}\right) = a \; or \; \theta - \emptyset = 2\cot^{-1} a$$

$$Thus \;\; \sin^{-1} x - \sin^{-1} y = 2 \;\cot^{-1} a \ldots\ldots (ii)$$

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}} . \frac{dy}{dx} = 0$$

$$\therefore \; \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

MOST PROBABLE QUESTIONS:

(1) Find the derivatives of y w.r.t. x in $x^2 + y^2$+5xy=0

(2) Find the derivative of y w.r.t. x in $sin^2 x + 2\cos y + xy = 0$

(3) Find $\frac{dy}{dx}, where \cos(x+y) = y \sin x$

(4) find $\frac{dy}{dx}, where \sin(xy) + \frac{x}{y} = x^2 - y$

(5) $find \; \frac{dy}{dx} \;, where \;\; (\cos x)^y = (\sin y)^x$

(6) Find $\frac{dy}{dx} \;, where \; y \cot x + y^3 \tan x + \sin x = 0$

Notes-17

TOPIC:DIFFERENTIATION 0f LOGARITHMIC

DESCRIPTION :

IF we are required to find the differential coefficient of a function whose power is a function of x , the standard result obtained so far can not be applied directly .in such case we first take the logarithmic of the function and then differentiate . this method is called the logarithmic differentiation.

when the given function is a power of some expression or a product of expression , we take logarithmic on both sides and differentiate the implicity function so obtained .

now we discuss some rules, by use these things to solve the standard problems.

(1) $y = [f(x)]^{g(x)}$
$ans: \log y = \log [f(x)]^{g(x)}$
$= g(x). \log f(x)$

(2) $y = [\, f(x). g(x)]$
$ans: \log y = \log[f(x). g(x)]$
$= \log f(x) + \log g(x)$

(3) $y = [f(x)]^{g(x)} + [h(x)]^{v(x)}$
$ans: let\ u = [f(x)]^{g(x)}$, v=$[h(x)]^{v(x)}$
$y = u + v$
$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

MODEL QUESTION:

(1) Find $\frac{dy}{dx}$ $, when\ y = x^x$
Ans: $y = x^x$ taking logarithm on both sides , we get
$\log y = \log x^x = x \log x$
Differentiating both sides with respect to x , we get
$\frac{d}{dx}(\log y) = \frac{d}{dx}(x \log x) = x. \frac{d}{dx}(\log x) + \log x. \frac{d}{dx}(x)$
$=> \frac{1}{y}\frac{dy}{dx} = x . \frac{1}{x} + \log x = 1 + \log x$
$=> \frac{dy}{dx} = (1 + \log x)y => \frac{dy}{dx} = x^x(1 + \log x)$

(2) Differentiate $(\tan x)^{secx}$

Ans: let $y = (\tan x)^{\sec x}$

Taking log on both sides

$\log y = \log(\tan x)^{\sec x}$

On differentiation ,

$\frac{1}{y} \cdot \frac{dy}{dx} = \sec x \cdot \frac{sec^2 x}{\tan x} + \sec x \tan x . \log (\tan x)$

$\frac{dy}{dx}$=$y[\frac{1}{\cos x} \cdot \frac{\cos x}{\sin x}. sec^2 x$+$\sec x . \tan x . \log (\tan x)]$

= $(\tan x)^{\sec x}$ $[cosec\ x. sec^2 x$+$\sec x . \tan x \log (tan\ x)]$

MOST PROBABLE QUESTIONS: Differentiate

(1) $(\sin x)^x$
(2) $x^{\sin^{-1} x}$
(3) $(\sin x)^{\log x}$
(4) Find the derivative of $(\cos x)^{\ln x} + (\log x)^x$
(5) Find the derivative of($\sin x)^{\cos^{-1} x}$
(6) Find the derivative of $(ax^2 + bx + c)^{\cos x}$
(7) If $\sqrt{1 - x^4} + \sqrt{1 - y^4} = k(x^2 - y^2) then\ show\ that\ \frac{dy}{dx} = \frac{x\sqrt{1-y^4}}{y\sqrt{1-x^4}}$
(8) If $x^y = e^{x-y}, prove\ that\ \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$
(9) Differentiate $\frac{e^{x^2} . \tan^{-1} x}{\sqrt{1+x^2}}$

Notes-18

TOPIC: DIFFERENTIATION 0f LOGARITHMIC

DESCRIPTION:

IF we are required to find the differential coefficient of a function whose power is a function of x , the standard result obtained so far can not be applied directly .in such case we first take the logarithmic of the function and then differentiate . this method is called the logarithmic differentiation.

when the given function is a power of some expression or a product of expression , we take logarithmic on both sides and differentiate the implicity function so obtained . here we discuss more problem related to logarithmic function.

now we discuss some rules, by use these things to solve the standard problems.

(4) $y = [f(x)]^{g(x)}$
$ans: \log y = \log [f(x)]^{g(x)}$
$= g(x). \log f(x)$

(5) $y = [\ f(x). g(x)]$
$ans: \log y = \log[f(x). g(x)]$
$= \log f(x) + \log g(x)$

(6) $y = [f(x)]^{g(x)} + [h(x)]^{v(x)}$
$ans: let\ u = [f(x)]^{g(x)}$, v=$[h(x)]^{v(x)}$
$y = u + v$
$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

MODEL QUESTION:

(1) If $y = x^{x^x}, find\ \frac{dy}{dx}$

Ans: let $y = x^{x^x}\ then \log y = x^x (\log x)$

On differentiating both sides with respect to x, we get :

$\frac{1}{y} . \frac{dy}{dx} = x^x . \frac{1}{x} + (\log x). \frac{d}{dx}\ (x^x)$

$\therefore \frac{dy}{dx} = y[x^{x-1} + (\log x). x^x (1 + \log x)]$

$[\therefore u = x^x => \log u = x \log x$

⇨ $\frac{1}{u} . \frac{du}{dx} = (x. \frac{1}{x} + (\log x. 1) => \frac{du}{dx} = u(1 + \log x) = x^x (1 + \log x)]$

$$hence\ , \frac{dy}{dx} = x^{x^x} [x^{x-1} + (\log x) x^x (1 + \log x)]$$

(2) Differentiate $\tan x \tan 2x \tan 3x \tan 4x$

Ans:

Let y= $\tan x \tan 2x \tan 3x \tan 4x$

Then , $\log y = \log(\tan x) + \log(\tan 2x) + \log(\tan 3x) + \log(\tan 4x)$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \{\frac{sec^2 x}{\tan x} + \frac{2sec^2 2x}{\tan 2x} + \frac{3sec^3 3x}{\tan 3x} + \frac{4sec^4 4x}{\tan 4x}\}$$

$$\therefore \frac{dy}{dx} = y[\frac{1}{\sin x \cos x} + \frac{2}{\sin 2x \cos 2x} + \frac{3}{\sin 3x \cos 3x} + \frac{4}{\sin 4x \cos 4x}]$$

= $y[\frac{2}{\sin 2x} + \frac{4}{\sin 4x} + \frac{6}{\sin 6x} + \frac{8}{\sin 8x}]$

= $[2 \tan x \tan 2x \tan 3x\ tan 4x] \times [cosec\ 2x + 2\ cosec\ 4x + 3\ cosec\ 6x + 4\ cosec\ 8x]$

MOST PROBABLE QUESTIONS: Differentiate

(1) x^x

(2) $(\ln x)^x$

(3) x^{x^2}

(4) $x^{\ln x}$

(5) $x^{\tan x} + \cos x^{\sin x}$

(6) $\sqrt{x(x+1)(x+2)}$

(7) $x^{\sin x} + \tan x^x$

(8) if $y = \left(\sqrt{x}^{\sqrt{x}^{\sqrt{x}}}\right), prove\ that\ (\frac{dy}{dx}) = \frac{y^2}{(2-ylogx)}$

(9) if $y = x^{x^{x^{\dots\dots\infty}}}, prove\ that\ \frac{dy}{dx} = \frac{y^2}{x(1-ylogx)}$

(10) if $y = \sqrt{sinx + \sqrt{\sin x + \cdots \infty}}$, prove that $\frac{dy}{dx} = \frac{\cos x}{(2y-1)}$

Notes-19

TOPIC:APPLICATION OF DERIVATIVE(Successive differentiation (second order)

DESCRIPTION:

Let $f(x)$ be a function, differentiable on an open interval (a,b) . then we know $f'(x)$ exit at each $x \in (a,b)$. The correspondence $x \rightarrow f'(x), x\epsilon(a,b)$ is a function in its own right . This new function is denoted by $f'(x)$ and is called derivative of $f(x)$.

If the derivative $f'(x)$ of a function $f(x)$ is itself differentiable then the derivative of $f'(x)$ is called the second order derivative of $f(x)$ and is denoted by $f''(x)$.

If $y = f(x)$ then $\frac{dy}{dx}$, the derivative of $y\ w.r.t\ x$, is itself , in general , a function of x and can be differentiable again. To fix up the idea , we shall call $\frac{dy}{dx}$ as the first order derivative of y with respect to x and the derivative of $\frac{dy}{dx}$ w.r.t x as second order derivative of y w.r.t x and will denoted by $\frac{d^2y}{dx^2}$ similarly the higher order derivative is denoted by $\frac{d^ny}{dx^n}$.

If $y = f(x)$ then the order alternative notation for $\frac{dy}{dx}, \frac{d^2x}{dx^2}$.........$\frac{d^nx}{dx^n}$. that is also denoted by $f'(x)$, $f''(x), \ldots \ldots \ldots . f^n(x)$. Which is denoted by $f_1, f_2 \ldots \ldots . . f_n$.

MODEL QUESTION:

(1) If $x = at^2, y = 2\ at\ find\ \frac{d^2y}{dx^2}$

Ans: Given $x = at^2$, $y = 2at$

$$\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a, \therefore \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = 2a.\frac{1}{2at} = \frac{1}{t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{1}{t}\right).\frac{dt}{dx} = -\frac{1}{t^2}.\frac{1}{2at} = -\frac{1}{2at^3}$$

(2) If $y = \tan^{-1} x$ $, prove\ that\ (1 + x^2)y_2 + 2xy_1 = 0$

Ans: Given $y = \tan^{-1} x$, $y_1 = \frac{1}{1+x^2}$

=>(1+x^2)y_1=1

Again differentiating w.r.t. x

- ⇨ (1+x^2).$\frac{d}{dx}(y_1) + y_1\frac{d}{dx}(1 + x^2) = 0$
- ⇨ (1+x^2)$y_2 + y_1.2x$=0
- ⇨ (1+x^2)$y_2 + 2xy_1 = 0$

(3) If y = Acosx+B sin nx then show that $\frac{d^2y}{dx^2} + n^2y = 0$

Ans: Given Acos nx + B sin nx

$$\frac{dy}{dx} = -A\sin nx\,.n + B\cos nx.n$$

$$\frac{d^2y}{dx^2} = -An.\cos nx.n - Bn.\sin nx.n = -n^2(A\cos nx + B\sin nx)$$

$\frac{d^2y}{dx^2} = -ny^2$ => $\frac{d^2y}{dx^2} + n^2y = 0$

MOST PROBABLE QUESTIONS:

(1) What is the slope of the curve y = sin x at x=$\frac{\pi}{6}$
(2) If y = x sin x , what is $y_1, at\ x = 0$
(3) Find the 2nd derivative of the function cos 2x .
(4) If x= 2 cos t – cos 2t ,y=2sint –sin2t, find $\frac{d^2y}{dx^2}$.
(5) If x= a$(\theta - sin\theta), y = a(1 + cos\theta), find\ \frac{dy}{dx}, \frac{d^2y}{dx^2}$
(6) If y= a sin 2x +b cos 2x , show that $\frac{d^2y}{dx^2}$+4y=0
(7) If y= sin (sin x) prove that
$\frac{d^2y}{dx^2} + tanx\frac{dy}{dx} + ycos^2x = 0$

(8) If Y = A cos nx +b sin nx then show that $\frac{d^2y}{dx^2} + n^2y = 0$.
(9) If y= $(\sin^{-1}x)^2$, show that (1-x^2)$y_2 - xy_1 - 2 = 0$.

Notes -20

TOPIC:PARTIAL DIFFERENTIAL EQUATION(Function of two variables second order)

DESCRIPTION: So far we have studied about derivatives of function of a single variable i.e y=(x)

$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x), in\ that\ case\ if\ a$ dependent variable is a function of single independent variable .in order to find the derivative of a function of two variables the following procedure is adopted

If a dependent variable is a function of two or more independent variables, in that case partial derivatives exists i.e z=f(x,y).The function is differentiated with respect to one of the independent variables while other is treated as constant.

Consider a function of two independent variables x and y. Let z=f(x,y) If the variable x under goes a change δx.let the variable y remains constant, then z undergoes a change δz

$$\delta z = f(x+\delta x, y) - f(x, y)$$

We say that z possess partial derivative w.r.t. x and denoted by

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x} = \frac{\partial f}{\partial x} = f_x$$

Similarly z possesses partial derivative w.r.t. y and denoted by

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} = \frac{\partial f}{\partial y} = f_y$$

Let z=f(x,y) be a function of two variables. Then
$\frac{\partial z}{\partial x} and\ \frac{\partial z}{\partial y} are\ themselves\ functions\ of\ two\ variables\ x\ and\ y, \frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q$

$=\frac{\partial^2 f}{\partial y^2} = f_{yy} = t\ \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = r\ \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

$\frac{\partial z^2}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{xy} = s, \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{yx} = s$

In general $\left(\frac{\partial^2 z}{\partial xy}\right) \neq \frac{\partial^2 z}{\partial y \partial x}$.

MODEL QUESTION:

(1) If z= $x^2y + xy^2$, $find\ \frac{\partial z}{\partial x} and\ \frac{\partial z}{\partial y}$

Ans: z=$x^2y + xy^2$

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2y + xy^2) = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial x}(xy^2)$=2xy+$y^2$

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}(x^2y+xy^2)=\frac{\partial}{\partial y}(x^2y)+\frac{\partial}{\partial y}(xy^2)=x^2+2xy$$

(2) If z=sin$\left(\frac{x}{y}\right)$ $find\,\frac{\partial z}{\partial x}\,and\,\frac{\partial z}{\partial y}$

Ans: Given z=sin$\left(\frac{x}{y}\right)$

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left\{\sin\left(\frac{x}{y}\right)\right\}=\cos\left(\frac{x}{y}\right)\frac{\partial}{\partial x}\left(\frac{x}{y}\right)=\frac{1}{y}\cos\left(\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left\{\sin\left(\frac{x}{y}\right)\right\}=\cos\left(\frac{x}{y}\right).\frac{\partial}{\partial y}\left(\frac{x}{y}\right)=\cos\left(\frac{x}{y}\right).\left(\frac{-x}{y^2}\right)=\frac{-x}{y^2}\cos\left(\frac{x}{y}\right)$$

(3) If $f(x,y)=\frac{2x-3y}{x^2+y^2}$, $find\ f_x(1,2) and\ f_y(1,2).$

Ans: $f(x,y)=\frac{2x-3y}{x^2+y^2}$. differentiate f w.r.t x , treating y as constant , we have

$$f_x(x,y)=\frac{(x^2+y^2).2-(2x-3y).2x}{(x^2+y^2)^2}=\frac{6xy-2x^2+2y^2}{(x^2+y^2)^2}$$

$$\therefore f_x(1,2)=\frac{6.1.2-2.1^2+2.2^2}{(1^2+2^2)^2}=\frac{18}{25}$$

Differentiating f w.r.t y, treating x as constant , we have

$$f_y(x,y)=\frac{(x^2+y^2).(-3)-(2x-3y).2y}{(x^2+y^2)^2}=\frac{3y^2-3x^2-4xy}{(x^{2+}y^2)^2}$$

$$\therefore f_y(1,2)=\frac{3.2^2-3.1^1-4.1.2}{(1^2+2^2)^2}=\frac{1}{25}$$

MOST PROBABLE QUESTIONS:

find f_x, f_y $where\ f(x,y)\ is\ given\ by$

(1) $\frac{x^2y+xy^2}{x+y}$

(2) x^y+y^x

(3) $\sin^{-1}\left(\frac{x}{y}\right)$

(4) If $f(x,y,z)=e^{xyz}$ then find $xf_x+yf_y+zf_z$

(5) If $u=(x^2+y^2+z^2)^{-\frac{1}{2}}$ show that $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}$=0

(6) If $z=f\left(\frac{y}{x}\right)$, $show\ that\ x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$

(7) Find the degree of the homogeneous function f(x,y) =$x^4+x^3y-y^4$,by two different methods.

(8) If z=$\tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$show that x$\frac{\partial z}{x}+\frac{\partial z}{\partial y}$= sin2z.

(9) If z =$\sin^{-1}\left(\frac{xy}{x+y}\right)$ show that x$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}$= tanx

Notes -21

TOPIC: PROBLEM BASED ON ABOVE

MOST PROBABLE QUESTIONS:

(1) IF $f(x, y) = e^{xy}$ what is $y\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial x}$

(2) If $u = \sin x \cos y,\ what\ is\ \frac{\partial u}{\partial y}$

(3) What is the slope of the curve $y = \sin x\ at\ x = \frac{\pi}{6}$

(4) What is the slope of the tangent to the curve $y = sinx\ at\ x = \frac{\pi}{3}$

(5) Find the slope of the curve $y = \frac{5}{3}x^2\ at\ x = 2$

(6) Find the derivative of the following functions

(i) $(x^3 + e^x + 3^x + \cot x)$

(ii) $(9x^2 + \frac{3}{x} + 5\sin x)$

(iii) $(x^2 + \frac{4}{x^2} - \frac{2}{3}\tan x + 7\ \log_e x + 6e$

(iv) $\log_e x^3$

(7) Given $y = (2x^3 - 4)^5$, find $\frac{dy}{dx}$

(8) Find the derivative of function

(i) $y = e^{\sin x}$

(ii) $y = \log(\sin x)$

(9) Find the derivative of $\sin^{-1} 5x$

(10) Find the derivative of $\tan^{-1}\sqrt{x}$

FIVE MARK QUESTION:

(1) Differentiate $\frac{(1-x)^{\frac{1}{2}}.(2-x^2)^{\frac{2}{3}}}{(3-x^3)^{\frac{3}{4}}.(4-x^4)^{\frac{4}{5}}}$

(2) Differentiate $e^{\sin^{-1} x}\ w.r.t\ e^{-\cos^{-1} x}$

(3) Differentiate $\ln(\sin x)\ w.r.t \tan x$

(4) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, find\ \frac{dy}{dx}$

(5) If sin $\sin y = x\sin(a + y)$, prove that $\frac{dy}{dx}$=$\frac{sin^2(a+y)}{\sin a}$

(6) Find the tangent line to $f(x) = 4\sqrt{2x} - 6e^{2-x}$ at x=2

(7) Find the derivative of $f(x) = \frac{1+e^{-2x}}{x+\tan(12x)}$ using chain rule.

(8) Find the derivative $h(u) = \tan(4 + 10u)$ by using chain rule.

(9) Find the derivative of $u(t) = \tan^{-1}(3t - 1)$ by using chain rule.

(10) Find the first derivative of $f(x) = (\sqrt{x} + 2x)(4x^2 - 1)$

INTEGRATION

Def^n:- After studying differentiation it is natural to study its Inverse process. This process is called Integration.

- **Antiderivative:-**

If $g(x)$ is the derivative of $f(x)$, then $f(x)$ is said to be antiderivative or integral of $g(x)$.

Ex:- $\frac{d}{dx}(\log x) = \frac{1}{x}$

i.e derivative of $\log x = \frac{1}{x}$

$\therefore$ Antiderivative of $\frac{1}{x} = \log x$ &

$$\int \frac{1}{x} \cdot dx = \log x.$$

- **Integral Calculus:-** The branch of calculus which studies about integration & its application is called Integral calculus.

→ Integral can be represented by summation. & also an elongated 'S' (for summation) is used to denote Integration.

- Again a constant always exists for an antiderivative

Ex:- $\frac{d}{dx}(\log x) = \frac{1}{x} + 0$

$\frac{d}{dx}(\log x + 1) = \frac{1}{x} \Rightarrow \int \frac{1}{x} \cdot dx = \log x + 1$

$\frac{d}{dx}(\log x + 5) = \frac{1}{x} \Rightarrow \int \frac{1}{x} \cdot dx = \log x + c$

$\therefore \int \frac{1}{x} \cdot dx = \log x + c$

So $\boxed{\int f(x)\cdot dx = F(x) + c}$

where $\int \rightarrow$ Symbol of Integration

$f(x)$ - any function x.

dx - integration with respect to x

$F(x)$ - Integral value

c - constant of Integration.

TYPES OF INTEGRALS :—

- Integrals are of Two types
  1) Indefinite Integral
  2) Definite Integral

1) Indefinite Integral :—

let $f(x)$ be a function. Then the family of all its primitives (or antiderivatives) is called indefinite Integral of $f(x)$ & it is denoted by $\int f(x).dx$.

i.e, $\frac{d}{dx}(\phi(x)+c) = f(x) \Leftrightarrow \int f(x).dx = \phi(x)+c$

2) Definite Integral :—

let $g(x)$ be the primitive or antiderivative of a continuous function $f(x)$ defined on $[a, b]$ i.e $\frac{d}{dx}\{g(x)\} = f(x)$

Then the definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x).dx = g(b) - g(a)$

where $a, b \rightarrow$ limits of Integration

b called Upper limit

a called lower limit

$[a, b] \rightarrow$ Interval of Integration

• **Application:-**

– Integration is used to find the areas under lines & curves. Entire area is divided into infinitesimally small regions, area of each region is found & added to get the entire area.

FUNDAMENTAL INTEGRAL FORMULAS:-

1. $\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c \quad , n \neq -1$

2. $\int \frac{1}{x} \cdot dx = \log_e|x| + c$

3. $\int e^x \cdot dx = e^x + c$

4. $\int a^x dx = \int \frac{a^x}{\log_e a} + c$

5. $\int \sin x \cdot dx = -\cos x + c$

6. $\int \cos x \cdot dx = \sin x + c$

7. $\int \sec^2 x \cdot dx = \tan x + c$

8. $\int \operatorname{cosec}^2 x \cdot dx = -\cot x + c$

9. $\int \sec x \cdot \tan x \cdot dx = \sec x + c$

10. $\int \operatorname{cosec} x \cdot \cot x \cdot dx = -\operatorname{cosec} x + c$

11. $\int \cot x \cdot dx = \ln|\sin x| + c$

12. $\int \tan x \cdot dx = -\ln|\cos x| + c$

13. $\int \sec x \cdot dx = \log|\sec x + \tan x| + c$

14. $\int \operatorname{cosec} x \cdot dx = \log|\operatorname{cosec} x - \cot x| + c$

15. $\int \frac{1}{\sqrt{a^2-x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

16. $\int \frac{-1}{\sqrt{a^2-x^2}} \cdot dx = \cos^{-1}\left(\frac{x}{a}\right) + c$

17. $\int \frac{1}{a^2+x^2} \cdot dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

18. $\int \frac{-1}{a^2+x^2} \cdot dx = \frac{1}{a} \cot^{-1}\left(\frac{x}{a}\right) + c$

19. $\int \frac{1}{x\sqrt{x^2-a^2}} \cdot dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$

20. $\int -\frac{1}{x\sqrt{x^2-a^2}} \cdot dx = \frac{1}{a} \operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + c$

21. $\int K \cdot dx = Kx + c$, K – constant

22. $\int \sqrt{x} \cdot dx = \frac{2}{3} x^{\frac{3}{2}} + c$

23. $\int \frac{1}{\sqrt{x}} \cdot dx = 2\sqrt{x} + c$

* Algebra of Integration:–

(i) $\int [f(x) + g(x)] \cdot dx = \int f(x) dx + \int g(x) \cdot dx$

(ii) $\int \lambda f(x) \cdot dx = \lambda \int f(x) \cdot dx$, for some constant λ.

(iii) $\int [\lambda_1 f_1(x) + \lambda_2 f_2(x) + \cdots + \lambda_n f_n(x)]\, dx$

$= \lambda_1 \int f_1(x) \cdot dx + \lambda_2 \int f_2(x) \cdot dx + \cdots + \lambda_n \int f_n(x) \cdot dx$

(iv) $\frac{d}{dx}\left(\int f(x)\cdot dx\right) = f(x)$.

i.e the differentiation of an integral is the integral itself or differentiation & integration are inverse operations.

• <u>problems</u>:—

<u>Ex:-1</u> $\int 4x^5\cdot dx = 4\int x^5\cdot dx = 4\left(\frac{x^{5+1}}{5+1}\right)+c = \frac{4}{6}x^6+c = \frac{2}{3}x^6+c$

<u>Eg.2</u>: $\int 2\sin x\cdot dx = 2\int \sin x\cdot dx = -2\cos x + c$

<u>Eg.3</u>. $\int 3^{x+2}\cdot dx = \int 3^x\cdot 3^2\cdot dx = 3^2\int 3^x\cdot dx$

$= 9\int 3^x\cdot dx$

$= 9\left(\frac{3^x}{\log 3}\right)+c$

<u>Eg.4</u>: $\int \frac{1}{2}\sec^2 x\cdot dx = \frac{1}{2}\int \sec^2 x\cdot dx = \frac{1}{2}\tan x + c$

<u>Eg.4</u>: $\int (x^6+x^2+x+1)\, dx$

$= \int x^6\cdot dx + \int x^2\cdot dx + \int x\, dx + \int 1\cdot dx$

$= \frac{x^{6+1}}{6+1} + \frac{x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1} + x + C \qquad \left[\because \int x^n\cdot dx = \frac{x^{n+1}}{n+1}+c\right]$

$= \frac{x^7}{7} + \frac{x^3}{3} + \frac{x^2}{2} + x + c$

<u>Eg.5</u> $\int e^{3x}\cdot dx = \int (e^3)^x\, dx = \frac{(e^3)^x}{\log e^3} + K = \frac{e^{3x}}{3} + K$

<u>Eg.6</u>: $\int \left(\frac{x^4}{x^2+1}\right)\cdot dx = \int\left(x^2 - 1 + \frac{1}{x^2+1}\right)dx$

$= \int x^2\cdot dx - \int 1\cdot dx + \int \frac{1}{x^2+1}\cdot dx$

$$= \frac{x^3}{3} - x + \tan^{-1}x + K$$

eg.7: $\int 6x^3(x+5)^2 \cdot dx$

$$= \int 6x^3(x^2+10x+25) \cdot dx$$

$$= \int (6x^5 + 60x^4 + 150x^3)\, dx$$

$$= \int 6x^5 \cdot dx + \int 60x^4\, dx + \int 150x^3 \cdot dx$$

$$= 6\int x^5 \cdot dx + 60\int x^4 \cdot dx + 150\int x^3 \cdot dx$$

$$= 6 \times \frac{x^{5+1}}{5+1} + 60 \cdot \frac{x^{4+1}}{4+1} + 150\, \frac{x^{3+1}}{3+1} + c$$

$$= \frac{\cancel{6}x^6}{\cancel{6}} + 60\frac{x^5}{5} + 150 \times \frac{x^4}{4} + C$$

$$= x^6 + 12x^5 + \frac{75}{2}x^4 + C$$

eg.8: $\int \frac{dx}{\cos^2 x} = \int -\sec^2 x \cdot dx = -\int \sec^2 x \cdot dx$

$$= -\tan x + C$$

eg.9: $\int (e^x+2)\,dx = \int e^x \cdot dx + \int 2 \cdot dx$

$$= e^x + 2x + C$$

eg.10 $\int \frac{x^4+x^3+x^2+x+2}{x^2+1} \cdot dx$

$$= \int \frac{(x^4+x^2)+(x^3+x)+2}{x^2+1} \cdot dx$$

$$= \int \frac{x^4+x^2}{x^2+1}\,dx + \int \frac{x^3+x}{x^2+1}\cdot dx + \int \frac{2}{x^2+1}\,dx$$

$$= \int \frac{x^4\cancel{(x^2+1)}}{\cancel{x^2+1}}\,dx + \int \frac{x\cancel{(x^2+1)}}{\cancel{x^2+1}}\,dx + 2\int \frac{1}{x^2+1}\,dx$$

$$= \int x^4\,dx + \int x\cdot dx + 2\int \frac{1}{x^2+1}\cdot dx$$

$$= \frac{x^{4+1}}{4+1} + \frac{x^{1+1}}{1+1} + 2\tan^{-1}x + c$$

$$= \frac{x^5}{5} + \frac{x^2}{2} + 2\tan^{-1}x + c$$

<u>eg 11</u>:- $\int \frac{1-\sin^3 x}{\sin^2 x}\cdot dx$

$$= \int \frac{1}{\sin^2 x}\cdot dx - \int \frac{\sin^{\cancel{3}} x}{\cancel{\sin^2 x}}\cdot dx$$

$$= \int \operatorname{cosec}^2 x\cdot dx - \int \sin x\cdot dx$$

$$= -\cot x + \cos x + C$$

<u>eg 12</u>:- $\int \sqrt{1-\cos 2x}\cdot dx$

$$= \int \sqrt{2\sin^2 x}\cdot dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right.$$

$$= \sqrt{2}\int \sqrt{\sin^2 x}\cdot dx$$

$$= \sqrt{2}\int \sin x\cdot dx$$

$$= \sqrt{2}\cos x + c$$

eg 13 :-

$$\int \frac{\cos^4 x - \sin^4 x}{\cos x - \sin x} \cdot dx$$

$$= \int \frac{(\cos^2 x)^2 - (\sin^2 x)^2}{\cos x - \sin x} \cdot dx$$

$$= \int \frac{(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)}{\cos x - \sin x} \cdot dx \quad \left[\because a^2 - b^2 = (a+b)(a-b)\right]$$

$$= \int \frac{1 \times (\cos^2 x - \sin^2 x)}{\cos x - \sin x} \cdot dx \quad \left[\because \cos^2 x + \sin^2 x = 1\right]$$

$$= \int \frac{(\cos x + \sin x)\cancel{(\cos x - \sin x)}}{\cancel{\cos x - \sin x}} \cdot dx \quad \left[\because a^2 - b^2 = (a+b)(a-b)\right]$$

$$= \int (\cos x + \sin x)\, dx$$

$$= \int \cos x \cdot dx + \int \sin x \cdot dx$$

$$= \int \cos x \cdot dx - \int -\sin x \cdot dx$$

$$= \sin x - \cos x + c$$

eg-14 :-

$$\int \frac{1 - \cos 2x}{1 + \cos 2x}\, dx$$

$$= \int \frac{\cancel{2}\sin^2 x}{\cancel{2}\cos^2 x} \cdot dx \quad \left[\begin{array}{l} \because \sin^2 x = \dfrac{1 - \cos 2x}{2} \\ \cos^2 x = \dfrac{1 + \cos 2x}{2} \end{array}\right]$$

$$= \int \tan^2 x \cdot dx$$

$$= \int (\sec^2 x - 1)\, dx$$

$$= \int \sec^2 x \cdot dx - \int 1 \cdot dx$$

$$= \tan x - x + c$$

eg-15 :> $\int \sqrt{1+\sin 2x}\,.dx$

$$= \int \sqrt{(\sin^2 x + \cos^2 x) + 2\sin x.\cos x}\; dx$$

$$= \int \sqrt{(\sin x + \cos x)^2}\,.dx$$

$$= \int \sin x + \cos x\,.dx$$

$$= \int \sin x\,dx + \int \cos x\,.dx$$

$$= -\cos x + \sin x + C$$

$$= \sin x - \cos x + C$$

eg-16 :> $\int \frac{\sin x}{\cos^2 x}\,.dx$

$$= \int \frac{\sin x}{\cos x.\cos x}\,.dx$$

$$= \int \tan x.\sec x.dx$$

$$= \sec x + C$$

eg-17 :> $\int (x^2 + \sqrt{x})^2\,.dx$

$$= \int \left(x^4 + (\sqrt{x})^2 + 2.x^2\sqrt{x}\right).dx$$

$$= \int x^4 + x + 2x^{5/2}.dx$$

$$= \int \left(x^4 + 2x^{5/2} + x\right) dx$$

$$= \int x^4.dx + \int 2x^{5/2}.dx + \int x.dx$$

$$= \frac{x^{4+1}}{4+1} + 2\frac{x^{5/2+1}}{5/2+1} + \frac{x^2}{2} + C$$

$$= \cancel{\frac{x^5}{5}} + \cancel{\frac{8x^{?}}{8}} + \cancel{\frac{x^2}{2}} + \cancel{C} = \frac{x^5}{5} +$$

METHODS OF INTEGRATION :—

We have the following Methods of Integration

(i) Integration by Substitution

(ii) Integration by parts

(iii) Integration of rational algebraic functions by using partial fractions.

(i) INTEGRATION BY SUBSTITUTION :—

When the integral is not in the standard form it can be transformed to integrable form by a suitable substitution. The Integral

$\int f(g(x))\, g'(x) \cdot dx$ can be converted to

$= \int f(t) \cdot dt$ where $g(x) = t$

$= F(t) + K$

There is no direct formula for substitution. Keen observation of the form of the Integrand will help choosing appropriate substitution.

Ex • $\int f(ax+b)\, dx$

let $ax+b = t$

$\Rightarrow d(ax+b) = dt$

$\Rightarrow a\,dx = dt$

$\Rightarrow dx = \frac{1}{a} dt$

substituting $ax+b=t$ & $dx=\frac{1}{a}dt$, we get

$$I=\int f(ax+b)\,dx=\int f(t)\,\frac{1}{a}\cdot dt=\frac{1}{a}\int f(t)\,dt$$

$$=\frac{1}{a}\Big/\frac{x^2}{2}=\frac{1}{a}\Big/ax+$$

$$=\frac{1}{a}\phi(t)=\frac{1}{a}\phi(ax+b)$$

Formula's Using Substitution: –

1. $\int(ax+b)^n dx=\dfrac{(ax+b)^{n+1}}{a(n+1)}+c\ ,\ n\neq-1$

2. $\int\dfrac{1}{ax+b}\cdot dx=\dfrac{1}{a}\log|ax+b|+c$

3. $\int e^{ax+b}\cdot dx=\dfrac{1}{a}e^{ax+b}+C$

4. $\int a^{bx+c}\cdot dx=\dfrac{1}{b}\dfrac{a^{bx+c}}{\log a}+c\ ,\ a>0\ \&\ a\neq1$

5. $\int\sin(ax+b)\,dx=\dfrac{-1}{a}\cos(ax+b)+c$

6. $\int\cos(ax+b)\,dx=\dfrac{1}{a}\sin(ax+b)+c$

7. $\int\sec^2(ax+b)\,dx=\dfrac{1}{a}\tan(ax+b)+c$

8. $\int\operatorname{cosec}^2(ax+b)\,dx=\dfrac{-1}{a}\cot(ax+b)+C$

9. $\int\sec(ax+b)\tan(ax+b)\,dx=\dfrac{1}{a}\sec(ax+b)+C$

10. $\int\operatorname{cosec}(ax+b)\cot(ax+b)\,dx=\dfrac{-1}{a}\operatorname{cosec}(ax+b)+c$

11. $\int\tan(ax+b)\,dx=\dfrac{1}{a}\log|\cos(ax+b)|+c$

12. $\int\cot(ax+b)\,dx=\dfrac{1}{a}\log|\sin(ax+b)|+c$

(13) $\int \sec(ax+b)\,dx = \frac{1}{a}\log|\sec(ax+b)+\tan(ax+b)| + c$

(14) $\int \operatorname{cosec}(ax+b)\,dx = \frac{1}{a}\log|\operatorname{cosec}(ax+b) - \cot(ax+b)| + c$

• <u>Problems</u> : —

① $\int e^{2x-3}\cdot dx = \frac{1}{2} \times e^{2x-3} + C$

② $\int e^{3x+2}\cdot dx = \frac{1}{3\log a} \times a^{3x+2} + c$

③ $\int \frac{\sin 4x}{\sin 2x}\cdot dx = \int \frac{2\sin 2x \cdot \cos 2x}{\sin 2x}\cdot dx$

$$= 2\int \cos 2x \cdot dx$$

$$= \frac{2}{2}\sin 2x + c$$

$$= \sin 2x + C$$

④ $\int \sqrt{1+\sin x}\cdot dx \quad , \quad 0 < x < \frac{\pi}{2}$

$$I = \int \sqrt{1+\sin x}\cdot dx$$

$$= \int \sqrt{\sin^2\frac{x}{2} + \cos^2 x/_2 + 2\sin x/_2 \cdot \cos x/_2}\cdot dx$$

$$= \int \sqrt{\left(\sin\frac{x}{2} + \cos x/_2\right)^2}\cdot dx$$

$$= \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) dx$$

$= \int \cos\frac{x}{2}\,.dx + \int \sin\frac{x}{2}\,.dx$

$= 2\sin\frac{x}{2} + 2\cos\frac{x}{2} + c$

$= 2\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + c$

Ex-5: $\int \frac{1}{\sqrt{3x+4} - \sqrt{3x+1}}$

Let $I = \int \frac{1}{\sqrt{3x+4} - \sqrt{3x+1}}\,.dx$

$= \int \frac{\sqrt{3x+4} + \sqrt{3x+1}}{\left(\sqrt{3x+4} - \sqrt{3x+1}\right)\left(\sqrt{3x+4} + \sqrt{3x+1}\right)}\,.dx$

$= \int \frac{\sqrt{3x+4} + \sqrt{3x+1}}{(3x+4) - (3x+1)}\,.dx$

$= \frac{1}{3}\int \left\{\sqrt{3x+4} + \sqrt{3x+1}\right\}.dx$

$= \frac{1}{3}\int \sqrt{3x+4}\,.dx + \frac{1}{3}\int \sqrt{3x+1}\,.dx$

$= \frac{1}{3}\left\{\frac{(3x+4)^{3/2}}{3\times\frac{3}{2}}\right\} + \frac{1}{3}\left\{\frac{(3x+1)^{3/2}}{3\times\frac{3}{2}}\right\} + c$

$= \frac{2}{27}\left\{(3x+4)^{3/2} + (3x+1)^{3/2}\right\} + c$

Ex-6: $\int \frac{8^{1+x} + 4^{1-x}}{2^x}\,.dx$

$I = \int \frac{8^{1+x} + 4^{1-x}}{2^x}\,.dx = \int \frac{2^{3x+3} + 2^{2-2x}}{2^x}\,.dx$

$= \int \left(2^{2x+3} + 2^{2-3x}\right)dx$

$= \frac{2^{2x+3}}{2\log 2} + \frac{2^{2-3x}}{(-3)\log 2} + c$

Ex. 7: $\int \sec^2(7-4x)\,dx = \frac{-1}{4}\tan(7-4x) + C$

#

Integration of Trigonometric function of the form

$$\int \sin mx \cdot \cos nx \cdot dx \;,\; \int \sin mx \cdot \sin nx \cdot dx \;,$$

$$\& \int \cos mx \cdot \cos nx \cdot dx$$

— To evaluate this type of integrals we use the following trigonometrical identities to express the products into sums

$$2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$$

• Problems:-

Ex-1: $\int \sin 4x \cos 3x \cdot dx$

$$I = \int \sin 4x \cdot \cos 3x \cdot dx$$
$$= \frac{1}{2}\int 2\sin 4x \cos 3x \cdot dx$$
$$= \frac{1}{2}\int (\sin 7x + \sin x)\, dx$$
$$= \frac{1}{2}\left\{ -\frac{\cos 7x}{7} - \cos x \right\} + c$$

Ex-2: $\int \sin 3x \cdot \sin 2x \cdot dx$

$$I = \int \sin 3x \cdot \sin 2x \, dx$$
$$= \frac{1}{2}\int 2\sin 3x \cdot \sin 2x \cdot dx$$

$$= \frac{1}{2}\int \{ \cancel{\mathrm{Scos}} \, (\cos x - \cos 5x) \, dx$$

$$= \frac{1}{2}\left\{ \sin x - \frac{\sin 5x}{5} \right\} + C$$

Ex-3: $I = \int \sin x \cdot \sin 2x \cdot \sin 3x \cdot dx$

$$= \frac{1}{2}\int (2\sin 2x \cdot \sin x)\sin 3x \cdot dx$$

$$\cancel{= \frac{1}{2}\int (\cos 6x + \cos 2x)\sin 3x \cdot dx}$$

$$= \frac{1}{2}\int (\cos x - \cos 3x)\sin 3x \cdot dx$$

$$= \frac{1}{2}\int (2\sin 3x \cos x - 2\sin 3x \cos 3x)\, dx$$

$$= \frac{1}{4}\int (\sin 4x + \sin 2x - \sin 6x)\, dx$$

$$= \frac{1}{4}\left\{ -\frac{\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6} \right\} + C$$

Ex-4 i) $I = \int \frac{\sin 4x}{\sin x} \cdot dx$

$$= \int \frac{2\sin 2x \cdot \cos 2x}{\sin x}\, dx$$

$$= \int \frac{4\cancel{\sin x} \cdot \cos x \cdot \cos 2x}{\cancel{\sin x}} \cdot dx$$

$$= \int 4\cos x \cdot \cos 2x \, dx$$

$$= 2\int 2\cos x \cdot \cos 2x \, dx$$

$$= 2\int (\cos 3x + \cos x)\, dx$$

$$= 2\left\{ \frac{\sin 3x}{3} + \sin x \right\} + c$$

$$= \frac{1}{2}\left\{ \frac{\sin 3x}{3} + \sin x \right\} + C$$

Ex-4 (i) $\int \sin^2 x \cdot dx$

$$= \int \frac{1}{2}(1 - \cos 2x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

$$= \frac{1}{4}\left(2x - \sin 2x\right) + C$$

Ex 5: $\int \sin^3 x \cdot \cos^3 x \cdot dx$

$$= \frac{1}{8}\int (2\sin x \cos x)^3 \cdot dx$$

$$= \frac{1}{8}\int (\sin 2x)^3\, dx$$

$$= \frac{1}{32}\int 4\sin^3 2x\, dx = \frac{1}{32}\int (3\sin 2x - \sin 6x)\, dx$$

$$= \frac{1}{32}\left[\frac{-3}{2}\cos 2x + \frac{1}{6}\cos 6x\right] + C$$

$$= \frac{1}{192}\left(\cos 6x - 9\cos 2x\right) + C$$

Evaluation of Integrals of the form $\int \sin^m x \cdot dx$, $\int \cos^m x\, dx$, $m \le 4$

- $\sin^2 x = \frac{1-\cos 2x}{2}$
- $\cos^2 x = \frac{1+\cos 2x}{2}$
- $\sin 3x = 3\sin x - 4\sin^3 x$
- $\cos 3x = 4\cos^3 x - 3\cos x$

Problems:-

Ex-1: $\int \sin^2 x \cdot dx$

$$= \int \frac{1-\cos 2x}{2} \cdot dx$$

$$= \frac{1}{2}\int 1-\cos 2x \cdot dx$$

$$= \frac{1}{2}\left\{ x - \frac{\sin 2x}{2}\right\} + C$$

Ex-2: $\int \sin^3 x \cdot dx$

$$= \int \frac{3\sin x - \sin 3x}{4} \cdot dx$$

$$= \frac{1}{4}\int (3\sin x - \sin 3x)\, dx$$

$$= \frac{1}{4}\left\{ -3\cos x + \frac{\cos 3x}{3}\right\} + c$$

Ex-3: $\int \sin^4 x \cdot \cos^4 x \cdot dx$

$$= \frac{1}{16}\int (2\sin x \cos x)^4\, dx$$

$$= \frac{1}{16}\int (\sin 2x)^4 \cdot dx$$

$$= \frac{1}{16}\int (\sin^2 2x)^2\, dx$$

$$= \frac{1}{16}\int \left(\frac{1-\cos 4x}{2}\right)^2 \cdot dx$$

$$= \frac{1}{64}\int (1 - 2\cos 4x + \cos^2 4x)\, dx$$

$$= \frac{1}{64}\int \left\{ 1 - 2\cos 4x + \frac{1+\cos 8x}{2}\right\} dx$$

$$= \frac{1}{128}\int (3 - 4\cos 4x + \cos 8x)\, dx$$

$$= \frac{1}{128}\left\{ 3x - \sin 4x + \frac{1}{8}\sin 8x\right\} + c$$

Evaluation of Integral of the form $\frac{P(x)}{(ax+b)^n}$, $n \in N$:—

Where $P(x)$ is a polynomial.

Problems:-

Ex-1:- $\int \frac{x^3}{(x+2)^4} \cdot dx$

$$= \int \frac{\{(x+2)-2\}^3}{(x+2)^4} \cdot dx$$

$$= \int \frac{(x+2)^3 - 8 - 6(x+2)^2 + 12(x+2)}{(x+2)^4} \cdot dx$$

$$= \int \left\{\frac{1}{x+2} - \frac{6}{(x+2)^2} + \frac{12}{(x+2)^3} - \frac{8}{(x+2)^4}\right\} \cdot dx$$

$$= \log|x+2| + \frac{6}{x+2} - \frac{6}{(x+2)^2} + \frac{8}{3(x+2)^3} + C$$

Ex-2:- $\int \frac{ax+b}{(cx+d)^2} \cdot dx$

Let $ax+b = \lambda(cx+d) + \mu$

On equating coefficient of like powers of x, we get $a = \lambda c$ & $b = \lambda d + \mu$.

$$\Rightarrow \lambda = \frac{a}{c}$$

$$\& \ \mu = \frac{bc - ad}{c}$$

$$= \int \frac{ax+b}{(cx+d)^2} \cdot dx$$

$$= \int \frac{\lambda(cx+d) + \mu}{(cx+d)^2} \cdot dx$$

$$= \lambda \int \frac{1}{cx+d}\cdot dx + \mu \int \frac{1}{(cx+d)^2}\cdot dx$$

$$= \frac{\lambda}{c} \log|cx+d| - \frac{\mu}{c(cx+d)} + C$$

$$= \frac{a}{c^2} \log|cx+d| - \frac{(bc-ad)}{c^2} \times \frac{1}{cx+d} + C$$

Ex-2: $\int \frac{x+2}{(x+1)^2}\cdot dx$

Let $x+2 = \lambda(x+1)+\mu$

On equating the coefficients of like powers of x on both sides, we get

$\lambda = 1$ & $2 = \lambda+\mu \Rightarrow \mu = 1$

$$= \int \frac{\lambda(x+1)+\mu}{(x+1)^2}\cdot dx$$

$$= \int \left\{\frac{\lambda}{x+1} + \frac{\mu}{(x+1)^2}\right\} dx$$

$$= \lambda \int \frac{1}{x+1}\, dx + \mu \int \frac{1}{(x+1)^2}\, dx$$

$$= \lambda \log|x+1| - \frac{\mu}{x+1} + C$$

$$= \log|x+1| - \frac{1}{x+1} + C$$

Ex-3: $\int \frac{x^2}{(a+bx)^2}\cdot dx$

using long division method

$$\frac{x^2}{(a+bx)^2} = \frac{1}{b^2} + \frac{-\frac{2a}{b}x - \frac{a^2}{b^2}}{(bx+a)^2}$$

$$\Rightarrow \frac{x^2}{(a+bx)^2} = \frac{1}{b^2} - \frac{a}{b^2}\left(\frac{2bx+a}{(bx+a)^2}\right)$$

$$\Rightarrow \frac{x^2}{(a+bx)^2} = \frac{1}{b^2} - \frac{a}{b^2}\left\{\frac{2(bx+a)-a}{(bx+a)^2}\right\}$$

$$= \frac{1}{b^2} - \frac{2a}{b^3} \times \frac{1}{bx+a} + \frac{a^2}{b^2} \times \frac{1}{(bx+a)^2}$$

$$\therefore \int \frac{x^2}{(a+bx)^2} \cdot dx$$

$$= \int\left\{\frac{1}{b^2} - \frac{2a}{b^3} \times \frac{1}{bx+a} + \frac{a^2}{b^2} \times \frac{1}{(bx+a)^2}\right\} \cdot dx$$

$$= \frac{1}{b^2}\int 1 \cdot dx - \frac{2a}{b^2}\int \frac{1}{bx+a} \cdot dx + \frac{a^2}{b^2}\int \frac{1}{(bx+a)^2} \cdot dx$$

$$= \frac{x}{b^2} - \frac{2a}{b^3}\log|bx+a| - \frac{a^2}{b^3} \times \frac{1}{bx+a} + C$$

$$= \frac{1}{b^3}\left\{bx - 2a\log|bx+a| - \frac{a^2}{bx+a}\right\} + C$$

Ex-9: $\int \frac{x^2+1}{(x+1)^2} \cdot dx$

$$= \int \frac{x^2+1+2x-2x}{(x+1)^2} \cdot dx$$

$$= \int \frac{(x+1)^2 - 2x}{(x+1)^2} \cdot dx$$

$$= \int 1 - \frac{2x}{(x+1)^2} \cdot dx$$

$$= \int 1 \cdot dx - 2\int \frac{x}{(x+1)^2}\, dx$$

$$= \int 1 \cdot dx - 2\int \frac{(x+1)-1}{(x+1)^2} \cdot dx$$

$= \int 1 \cdot dx - 2\int \left\{ \frac{1}{x+1} - \frac{1}{(x+1)^2} \right\} dx$

$= \int 1 \cdot dx - 2\int \frac{1}{x+1} \cdot dx + 2\int \frac{1}{(x+1)^2}\, dx$

$= x - 2\log|x+1| - \frac{2}{x+1} + c$

Evaluation of Integral of the form

$\int (ax+b)\sqrt{cx+d} \cdot dx$ & $\int \frac{ax+b}{\sqrt{cx+d}} \cdot dx$:—

Algorithm:—

Step-I: Let $(ax+b) = \lambda(cx+d) + \mu$

step-II: find λ & μ equating the coefficients of like powers of x on both sides.

step-III: Replace $(ax+b)$ by $\lambda(cx+d)+\mu$ ~~we get~~ & integrating we get the value.

Problems:-

Ex-1: $\int x\sqrt{x+2} \cdot dx$

$= \int \{(x+2)-2\}\sqrt{x+2} \cdot dx$

$= \int \{(x+2)^{3/2} - 2(x+2)^{1/2}\} dx$

$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c$

Ex-2: $\int (7x-2)\sqrt{3x+2} \cdot dx$

Let $7x - 2 = \lambda(3x+2) + \mu$

$3\lambda = 7 \Rightarrow \mu = -\frac{20}{3}, \lambda = 7/3$

$= \int \{\lambda(3x+2)+\mu\}\sqrt{3x+2} \cdot dx$

$$= \int \left\{ \lambda(3x+2)^{3/2} + \mu(3x+2)^{1/2} \right\} . dx$$

$$= \lambda \left\{ \frac{(3x+2)^{5/2}}{5/2 \times 3} \right\} + \mu \left\{ \frac{(3x+2)^{3/2}}{3 \times \frac{3}{2}} \right\} + c$$

$$= \frac{14}{45}(3x+2)^{5/2} - \frac{40}{27}(3x+2)^{3/2} + C$$

Ex/12:-

Integral of the form $\int \frac{f'(x)}{f(x)} . dx$:—

Theorem :- $\int \frac{f'(x)}{f(x)} . dx = \log\{f(x)\} + C$

Problems:-

① $\int \tan x . dx = \int \frac{\sin x}{\cos x} . dx$

$\cos x = t$ & $dx = -dt/\sin x$

$$I = \int \frac{\sin x}{\cos x} \times \left(-\frac{dt}{\sin x}\right)$$

$$= -\int \frac{1}{t} dt = -\log|t| + c$$

$$= -\log|\cos x| + C$$

$$\boxed{\int \tan x . dx = -\log|\cos x| + c}$$

Results—

i) $\int \tan x \, dx = -\log|\cos x| + C$

ii) $\int \cot x . dx = \log|\sin x| + C$

iii) $\int \sec x . dx = \log|\sec x + \tan x| + C$

&

$\log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(iv) $\int \operatorname{cosec} x \cdot dx = \log|\operatorname{cosec} x - \cot x| + C$ & $\log\left|\tan \frac{x}{2}\right| + C$

(v)

Problem:-

① $\int \frac{1}{\sqrt{1+\cos 2x}} \cdot dx$

$= \int \frac{1}{\sqrt{2\cos^2 x}} \cdot dx$

$= \frac{1}{\sqrt{2}} \int \sec x \cdot dx = \frac{1}{\sqrt{2}} \log|\sec x + \tan x| + c$

② $\int \frac{1}{\sqrt{1-\sin x}} \cdot dx$

$= \int \frac{1}{1-\cos(\pi/2 + 2x)} \cdot dx$

$= \int \frac{1}{2\sin^2(\pi/2 + x)} \cdot dx$

$= \int \frac{1}{\sqrt{2}} \operatorname{cosec}\left(\frac{\pi}{4} + x\right) \cdot dx$

$= \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(\frac{\pi}{4} + x\right) dx$

$= \frac{1}{\sqrt{2}} \log\left|\tan\left(\frac{\pi}{8} + \frac{x}{2}\right)\right| + C$

③ $\int \frac{\sin(x-a)}{\sin x} \cdot dx$

$= \int \frac{\sin x \cos a - \cos x . \sin a}{\sin x} \cdot dx$

$= \int \cos a \cdot dx - \int \sin a \cot x \cdot dx$

$= \cos a \int 1 \cdot dx - \sin a \int \cot x \cdot dx$

$= x \cos a - \sin a \log|\sin x| + c$

Ex-2: $\int \frac{1}{\sin(x-a)\cdot\cos(x-b)}\cdot dx$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}\cdot dx$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cdot\cos(x-b)}\cdot dx$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos(x-a)\cos(x-b) + \sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}\cdot dx$$

$$= \frac{1}{\cos(a-b)} \int \{\cot(x-a) + \tan(x-a)\} dx$$

$$= \frac{1}{\cos(a-b)} \int \cot(x-a) + \tan(x-b)\}\cdot dx$$

$$= \frac{1}{\cos(a-b)} \{ \log_e|\sin(x-a)| - \log_e|\cos(x-b)|\} + c$$

$$= \frac{1}{\cos(a-b)} \log_e\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + c$$

Ex-3:- $\int \frac{2x+5}{x^2+5x-7}\cdot dx$

Let $x^2+5x-7 = t$, then $d(x^2+5x-7) = dt$

$\Rightarrow (2x+5)dx = dt$

$\Rightarrow dx = \frac{dt}{2x+5}$

Putting $x^2+5x-7 = t$ & $dx = \frac{dt}{2x+5}$, we get

$$I = \int \frac{2x+5}{x^2+5x-7}\, dx = \int \frac{1}{t}\, dt = \log|t| + C$$

$$= \log|x^2+5x-7| + C$$

Ex-4: $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

Let $e^x + e^{-x} = t$

$\Rightarrow (e^x - e^{-x})\, dx = dt$

$\Rightarrow \quad dx = \frac{dt}{e^x - e^{-x}}$

$$= \int \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot dx = \int \frac{dt}{t} = \log|t| + C = \log|e^x + e^{-x}| + C$$

Integration of the form $\int \{f(x)\}^n f'(x)\cdot dx$:—

Theorem: — $\int \{f(x)\}^n f'(x)\cdot dx = \frac{\{f(x)\}^{n+1}}{n+1}, \; n \neq -1$

· Problems:

(1) $\int \sin^3 x \cos x \cdot dx$

Let $\sin x = t$, then $d(\sin x) = dt$

$\Rightarrow \cos x \cdot dx = dt$

$\Rightarrow dx = \frac{dt}{\cos x}$

Putting $\sin x = t$ & $dx = \frac{dt}{\cos x}$, we get

$$\int \sin^3 x \cos x \cdot dx = \int t^3 \cancel{\cos x} \times \frac{dt}{\cancel{\cos x}} = \int t^3 dt$$

$$= \frac{t^4}{4} + C = \frac{\sin^4 x}{4} + C$$

(2) $\int \frac{4(\sin^{-1}x)^3}{\sqrt{1-x^2}} \cdot dx$

Let $\sin^{-1}x = t$

$\Rightarrow d(\sin^{-1}x) = dt$

$\Rightarrow \frac{1}{\sqrt{1-x^2}} \cdot dx = dt$

$\Rightarrow dx = \sqrt{1-x^2} \cdot dt$

Putting the above values in the integration we get.

$\int 4t^3 \cdot dt = 4\int t^3 dt = 4 \times \frac{t^4}{4} + c = t^4 + C = (\sin^{-1}x)^4 + C$

(31)

Integration of the form $\int \tan^m x \sec^{2n} x \cdot dx$, $\int \cot^m x \operatorname{cosec}^{2n} x \cdot dx$:-

Ex:- $\int \tan^3 x \sec^3 x \cdot dx$

$= \int \tan^2 x \cdot \sec^2 x \cdot (\sec x \cdot \tan x) \cdot dx$

$= \int (\sec^2 x - 1) \sec^2 x \, (\sec x \cdot \tan x) \cdot dx$

Now substituting $\sec x = t$ & $\sec x \cdot \tan x \, dx = dt$, we get

$= \int (t^2 - 1) t^2 \cdot dt = \int (t^4 - t^2) dt$

$= \frac{t^5}{5} - \frac{t^3}{3} + c$

$= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$

Integral of the form $\int \sin^m x \cos^n x \cdot dx$, $m, n \in N$

Ex:- (1) $\int \sin^3 x \cos^4 x \cdot dx$

Here power of $\sin x$ is odd, So, we substitute

$\cos x = t$

$\Rightarrow -\sin x \cdot dx = dt$

$\Rightarrow dx = -\dfrac{dt}{\sin x}$

$= \int \sin^3 x \; t^4 \left(-\dfrac{dt}{\sin x}\right)$

$= -\int \sin^2 x \cdot t^4 \, dt$

$= -\int (1-t^2) t^4 \, dt$

$= -\int (t^4 - t^6) \, dt$

$= -\dfrac{t^5}{5} + \dfrac{t^7}{7} + c$

$= -\dfrac{\cos^5 x}{5} + \dfrac{\cos^7 x}{7} + C$

(2) $\int \sin^2 x \cdot \cos^5 x \cdot dt$

Here $\cos x$ is odd, so,

let $\sin x = t$

$\Rightarrow \cos x \cdot dx = dt$

$\Rightarrow dx = \dfrac{dt}{\cos x}$

$= \int t^2 \cos^5 x \cdot \dfrac{dt}{\cos x} = \int t^2 (1-\sin^2 x)^2 \, dt$

$= \int t^2 (1-t^2)^2 \, dt$

$= \int (t^2 - 2t^4 + t^6) \, dt$

$= \frac{t^3}{3} - \frac{2}{5}t^5 + \frac{t^7}{7} + c$

$= \frac{\sin^3 x}{3} - \frac{2}{5}\sin^5 x + \frac{\sin^7 x}{7} + c$

(3) $\int \frac{\sin^4 x}{\cos^8 x} \cdot dx$

$= \int \frac{\frac{\sin^4 x}{\cos^4 x}}{\frac{\cos^8 x}{\cos^4 x}} dx$ [Devide numerator & denominator by $\cos^4 x$]

$= \int \tan^4 x \cdot \sec^4 x \cdot dx$

$= \int \tan^4 x \, (1 + \tan^2 x) \sec^2 x \cdot dx$

$= \int \tan^4 x \, (1 + \tan^2 x) \sec^2 x \cdot dx$

$\tan x = t \Rightarrow \sec^2 x \cdot dx = dt$

$= \int t^4 (1 + t^2) dt$

$= \frac{t^5}{5} + \frac{t^7}{7} + c$

$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$

Evaluation of Integrals By Using Trigonometric Substitution :—

Expression	Substitution
a^2+x^2	$x = a\tan\theta$ or $a\cot\theta$
a^2-x^2	$x = a\sin\theta$ or $a\cos\theta$
x^2-a^2	$x = a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ or, $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

Ex: $\int \frac{1}{(a^2-x^2)^{3/2}}\,dx$

Let $x = a\sin\theta$

$\Rightarrow dx = d(a\sin\theta)$

$\Rightarrow dx = a\cos\theta \cdot d\theta$

$$= \int \frac{1}{(a^2-a^2\sin^2\theta)^{3/2}}\, a\cos\theta \cdot d\theta$$

$$= \int \frac{a\cos\theta}{a^3\cos^3\theta}\, d\theta$$

$$= \frac{1}{a^2}\int \sec^2\theta \cdot d\theta$$

$$= \frac{1}{a^2}\tan\theta + C = \frac{1}{a^2}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{x}{a^3\left(\sqrt{1-\frac{x^2}{a^2}}\right)} + C$$

$$= \frac{x}{a^2\sqrt{a^2-x^2}} + C$$

(2) $\int \frac{x^2}{\sqrt{1-x}} \cdot dx$

$= \int \frac{x^2}{\sqrt{1-(\sqrt{x})^2}} \cdot dx$

Let $\sqrt{x} = \sin\theta$

$\Rightarrow x = \sin^2\theta$

$dx = d(\sin^2\theta)$

$= 2\sin\theta \cdot \cos\theta \cdot d\theta$

$= \int \frac{(\sin^2\theta)^2}{\sqrt{1-\sin^2\theta}} \cdot 2\sin\theta \cdot \cos\theta \cdot d\theta$

$= 2\int \sin^5\theta \cdot d\theta$

$= 2\int (1-\cos^2\theta)^2 \sin\theta \cdot d\theta$

Let $\cos\theta = u$

$\Rightarrow -\sin\theta \cdot d\theta = du$

$= -2\int (1-u^2)^2 du = -2\int (1-2u^2+u^4) du$

$= -2\left(u - \frac{2}{3}u^3 + \frac{u^5}{5}\right) + C$

$= \frac{-2}{15} u (15 - 10u^2 + 3u^4) + C$

$= \frac{-2}{15}(15 - 10\cos^2\theta + 3\cos^4\theta)\cos\theta + C$

$= \frac{-2}{15}\left\{15 - 10(1-\sin^2\theta) + 3(1-\sin^2\theta)^2\right\}\sqrt{1-\sin^2\theta} + C$

$= \frac{-2}{15}\left\{8 + 4\sin^2\theta + 3\sin^4\theta\right\}\sqrt{1-\sin^2\theta} + C$

$= \frac{-2}{15}\left\{8 + 4\sin^2\theta + 3\sin^4\theta\right\}\sqrt{1-\sin^2\theta} + C$

$= \frac{-2}{15}(8 + 4x + 3x^2)\sqrt{1-x} + C$

Theorem :-

(i) $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

(ii) $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

(iii) $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(iv) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

(v) $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left|x+\sqrt{a^2+x^2}\right| + c$

(vi) $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right| + c$

Ex: ① $\int \frac{1}{4+9x^2}\,dx$

$= \frac{1}{9}\int \frac{1}{\frac{4}{9}+x^2}\,dx$

$= \frac{1}{9}\int \frac{1}{\left(\frac{2}{3}\right)^2+x^2}\,dx$

$= \frac{1}{9}\times\frac{1}{(2/3)}\tan^{-1}\left(\frac{x}{2/3}\right) + C$

$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + C$

② $\int \frac{1}{9x^2-4}\,dx$

$= \frac{1}{9}\int \frac{1}{x^2-\left(\frac{2}{3}\right)^2}\,dx = \frac{1}{9}\times\frac{1}{2\times\frac{2}{3}}\log\left|\frac{x-\frac{2}{3}}{x+\frac{2}{3}}\right| = \frac{1}{12}\log\left|\frac{3x-2}{3x+2}\right| + c$

(3) $\int \frac{1}{16-9x^2} dx$

$= \frac{1}{9}\int \frac{1}{\frac{16}{9} - x^2} \cdot dx$

$= \frac{1}{9}\int \frac{1}{\left(\frac{4}{3}\right)^2 - (x)^2} dx = \frac{1}{9} \times \frac{1}{2\times\frac{4}{3}} \times \log\left|\frac{\frac{4}{3}+x}{\frac{4}{3}-x}\right| + C$

$= \frac{1}{24}\log\left|\frac{4+3x}{4-3x}\right| + C$

(4) $\int \frac{1}{\sqrt{9-25x^2}} \cdot dx$

$= \frac{1}{5}\int \frac{1}{\sqrt{\frac{9}{25} - x^2}} \cdot dx$

$= \frac{1}{5}\int \frac{1}{\sqrt{\left(\frac{3}{5}\right)^2 - x^2}} \cdot dx$

$= \frac{1}{5}\sin^{-1}\left(\frac{x}{3/5}\right) + C = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$

(5) $\int \frac{1}{x^2-x+1} \cdot dx$

$= \int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+1} dx$

$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} \cdot dx$

$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$

$$= \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{x - 1/2}{\sqrt{3}/2}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c$$

(6) $$\int \frac{1}{4x^2 - 4x + 3} \cdot dx$$

$$= \frac{1}{4} \int \frac{1}{x^2 - x + 3/4} \cdot dx$$

$$= \frac{1}{4} \int \frac{1}{x^2 - x + \frac{1}{4} - \frac{1}{4} + \frac{3}{4}} \cdot dx$$

$$= \frac{1}{4} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \cdot dx$$

$$= \frac{1}{4} \times \frac{1}{(1/\sqrt{2})} \tan^{-1}\left(\frac{x - 1/2}{1/\sqrt{2}}\right) + c$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + c$$

(7) $$\int \frac{1}{\sqrt{(x-1)(x-2)}} \cdot dx$$

$$= \int \frac{1}{\sqrt{x^2 - 3x + 2}} \cdot dx$$

$$= \int \frac{1}{\sqrt{x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2}} \cdot dx$$

$$= \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} \cdot dx$$

$$= \log\left|\left(x - \frac{3}{2}\right) + \sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + c$$

$$= \log\left|\left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2}\right| + c$$

(8) $\int \sqrt{\sec x - 1}\, dx$

$= \int \sqrt{\frac{1-\cos x}{\cos x}}\, dx$

$= \int \sqrt{\frac{1-\cos x}{\cos x} \times \frac{(1+\cos x)}{(1+\cos x)}}\, dx$

$= \int \sqrt{\frac{1-\cos^2 x}{\cos x + \cos^2 x}}\, dx$

$= \int \frac{\sin x}{\sqrt{\cos^2 x + \cos x}}$

Let $\cos x = t$

$\Rightarrow -\sin x \cdot dx = dt$

$\Rightarrow dx = \frac{-dt}{\sin x}$

$= \int \frac{-dt}{\sqrt{t^2+t}} = -\int \frac{dt}{\left(t+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$

$= -\log\left|\left(t+\frac{1}{2}\right) + \sqrt{\left(t+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} + C\right.$

$= -\log\left|\left(t+\frac{1}{2}\right) + \sqrt{t^2+t}\right| + C$

$= -\log\left|\left(\cos x + \frac{1}{2}\right) + \sqrt{\cos^2 x + \cos x}\right| + C$

(9) $\int \frac{x+5}{\sqrt{x^2+6x-7}} \cdot dx$

$= \int \frac{x+3+2}{\sqrt{(x+3)^2-16}} \cdot dx$

$= \int \frac{z+2}{\sqrt{z^2-16}}\, dz$, putting $x+3 = z$
$\Rightarrow dx = dz$

$$= \int \frac{z}{\sqrt{z^2-16}}\,dz + 2\int \frac{dz}{\sqrt{z^2-16}}$$

$$= \frac{1}{2}\int \frac{d(z^2-16)}{\sqrt{z^2-16}} + 2\int \frac{dz}{\sqrt{z^2-16}}$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{u}} + 2\int \frac{dz}{\sqrt{z^2-16}}$$

$$= \sqrt{u} + 2\ln\left|z + \sqrt{z^2-16}\right| + C$$

$$= \sqrt{z^2-16} + 2\ln\left|z + \sqrt{z^2-16}\right| + C$$

$$= \sqrt{(x+3)^2-16} + 2\ln\left|x+3+\sqrt{(x+3)^2-16}\right| + C$$

$$= \sqrt{x^2+6x-7} + 2\ln\left|(x+3) + \sqrt{x^2+6x-7}\right| + C$$

Integration By Parts:-

If u & v are two functions of x, then

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

i.e. The Integral of the product of of two functions = (first function) × (Integral of second function) − Integral of {(Differentiation of first function) × (Integral of second function)}

ILATE Rule:—

The first function can be choose using ILATE Rule where
I = Inverse Trigonometric fun^n
L = Logarithmic fun^n
A = Algebraic function
T = Trigonometric function
E = Exponential function

Problems:—

① $\int \underset{\text{I}}{x}\, \underset{\text{II}}{\sin 3x}\,dx$

$$= x\left\{\int \sin 3x\,dx\right\} - \int\left\{\frac{d}{dx}(x) \times \int \sin 3x\,dx\right\}dx$$

$$= x \times \frac{-1}{3}\cos 3x - \int\left\{\frac{-1}{3}\cos 3x\right\}.dx$$

$$= \frac{-1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx$$

$$= \frac{-1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$$

(2) $\int x \log x \, dx$

$= \log x \left\{ \int x \cdot dx \right\} - \int \left\{ \frac{d}{dx}(\log x) \times \int x \, dx \right\} dx$

$= (\log x) \frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2} \, dx$

$= \frac{x^2}{2} \log x - \frac{1}{2} \int x \cdot dx$

$= \frac{x^2}{2} \log x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C$

$= \frac{x^2}{2} \log x - \frac{1}{4} x^2 + C$

(3) $\int \log x \cdot dx$

$= \int \underset{\mathrm{I}}{\log x} \cdot \underset{\mathrm{II}}{1} \cdot dx$

$= \log x \left\{ \int 1 \cdot dx - \int \frac{1}{\cancel{x}} \cdot x \cdot dx \right.$

$= x \log x - x + C$

(4) $\int e^x (\sin x + \cos x) \, dx$

$= \int \underset{\mathrm{II}}{e^x} \underset{\mathrm{I}}{\sin x} \cdot dx + \int e^x \cos x \cdot dx$

$= (\sin x) e^x - \int \cancel{\cos x \, e^x \cdot dx} + \int \cancel{e^x \cos x} \, dx + C$

$= e^x \sin x + C$

(5) $\int \sin^{-1}x \cdot dx$

Let $\int \sin^{-1}x \cdot dx = t$

Then $x = \sin t$

$\Rightarrow dx = \cos t \cdot dt$

$I = \int \underset{I}{t} \underset{II}{\cos t} \cdot dt$

$= t\sin t - \int 1 \cdot \sin t \cdot dt$

$= t\sin t - \int \sin t \cdot dt$

$= t\sin t + \cos t + C$

$= x\sin^{-1}x + \sqrt{1-\sin^2 t} + C$

$= x\sin^{-1}x + \sqrt{1-x^2} + C$

(6) $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$

$= \int e^x \cdot \frac{1}{x} \cdot dx - \int e^x \cdot \frac{1}{x^2} \cdot dx$

$= \frac{1}{x}e^x - \int \frac{-1}{x^2}e^x \cdot dx - \int e^x \times \frac{1}{x^2} \cdot dx + C$

$= \frac{1}{x}e^x + \cancel{\int \frac{1}{x^2}e^x \cdot dx} - \cancel{\int \frac{1}{x^2}e^x \cdot dx} + C$

$= \frac{1}{x}e^x + C$

SOME IMPORTANT INTEGRALS :—

Theorem

(i) $\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)+C$

(ii) $\int\sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}a^2\log|x+\sqrt{a^2+x^2}|+C$

(iii) $\int\sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\log|x+\sqrt{x^2-a^2}|+C$

· Problems :

(1) $\int\sqrt{4x^2+9}\,dx$

$= 2\int\sqrt{x^2+\frac{9}{4}}\,dx$

$= 2\int\sqrt{x^2+\left(\frac{3}{2}\right)^2}\,dx$

$= 2\left\{\frac{1}{2}x\sqrt{x^2+\frac{9}{4}}+\frac{1}{2}\left(\frac{3}{2}\right)^2\log\left|x+\sqrt{x^2+\frac{9}{4}}\right|\right\}+C$

$= \frac{x}{2}\sqrt{4x^2+9}+\frac{9}{4}\log|2x+\sqrt{4x^2+9}|+C$

(2) $\int\frac{x^2}{\sqrt{1-2x-x^2}}\,dx$

$= -\int\frac{-x^2}{\sqrt{1-2x-x^2}}\,dx = -\int\frac{(1-2x-x^2)+(2x-1)}{\sqrt{1-2x-x^2}}\,dx$

$= -\int\sqrt{1-2x-x^2}\,dx - \int\frac{2x-1}{\sqrt{1-2x-x^2}}\,dx$

$= -\int\sqrt{1-2x-x^2}\,dx + \int\frac{-2x-2+3}{\sqrt{1-2x-x^2}}\,dx$

$$= -\int\sqrt{1-2x-x^2}\,dx + \int\frac{-2x-2}{\sqrt{1-2x-x^2}}\,dx + 3\int\frac{1}{\sqrt{1-2x-x^2}}\,dx$$

$$= -\int\sqrt{(\sqrt{2})^2-(x+1)^2}\,dx + \int\frac{1}{\sqrt{1-2x-x^2}}\,d(1-2x-x^2)$$
$$+ 3\int\frac{1}{\sqrt{(\sqrt{2})^2-(x+1)^2}}\,dx$$

$$= \frac{-1}{2}\left\{(x+1)\sqrt{1-2x-x^2} + 2\sin^{-1}\frac{x+1}{\sqrt{2}}\right\} + 2\sqrt{1-2x-x^2} + 3\sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$
$$+C$$

(3) $\int(x-5)\sqrt{x^2+x}\,dx$

let $x-5 = \lambda\frac{d}{dx}(x^2+x)+\mu$

$\Rightarrow x-5 = \lambda(2x+1)+\mu$

comparing coefficients of like powers of x, we get,

$1 = 2\lambda$ & $\lambda+\mu = -5 \Rightarrow \lambda = \frac{1}{2}$ & $\mu = \frac{-11}{2}$

$$= \int(x-5)\sqrt{x^2+x}\,dx$$

$$= \int\left(\frac{1}{2}(2x+1)-\frac{11}{2}\right)\sqrt{x^2+x}\,dx$$

$$= \int\frac{1}{2}(2x+1)\sqrt{x^2+x}\,dx - \frac{11}{2}\int\sqrt{x^2+x}\,dx$$

$$= \frac{1}{2}\int\sqrt{t}\,dt - \frac{11}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx \quad \text{where } t = x^2+x$$

$$= \frac{1}{2}\times\frac{t^{3/2}}{3/2} - \frac{11}{2}\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right.$$
$$\left. - \frac{1}{2}\left(\frac{1}{2}\right)^2\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|\right\} + C$$

$$= \frac{1}{3}t^{3/2} - \frac{11}{2}\left\{ \frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x}\right|\right\} + c$$

$$= \frac{1}{3}\left(x^2+x\right)^{3/2} - \frac{11}{2}\left\{ \frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\log\left|\left(x+\frac{1}{2}\right)\sqrt{x^2+x}\right|\right\}$$

INTEGRATION OF RATIONAL ALGEBRAIC FUNCTIONS BY USING PARTIAL FRACTIONS ;—

if $f(x)$ & $g(x)$ are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function or a rational function of x.

(i) if degree of $f(x) <$ degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called a proper rational function.

(ii) if degree of $f(x) \geq$ degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called Improper rational function.

— For improper rational function, we divide $f(x)$ by $g(x)$ & expressed as

$$\frac{f(x)}{g(x)} = \phi(x) + \frac{\psi(x)}{g(x)}$$

Where $\phi(x)$ & $\psi(x)$ are polynomials such that the degree of $\psi(x)$ is less than that of $g(x)$., then it reduce to proper rational function.

→ For proper fraction $\frac{f(x)}{g(x)}$ can be decomposed into simpler fractions, called partial fractions & each simpler fractions integrated separately.

→ 4 types of different cases arises depending on denominator $Q(x)$.

(i) if $g(x)$ is non-repeating factor, i.e

$$g(x) = (a_1x+b_1)(a_2x+b_2)(a_3x+b_3)$$

$$\therefore \frac{f(x)}{g(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \frac{A_3}{a_3x+b_3}$$

(ii) if $g(x)$ has ~~so~~ some repeating factor,

let, $g(x) = (ax+b)^3(cx+d)$

$$\therefore \frac{f(x)}{g(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \frac{A_4}{(cx+d)}$$

(iii) for a non-repeated quadratic factor lx^2+px+q,

if $g(x) = (lx^2+px+q)(ax+b)$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{A_1x+B_1}{lx^2+px+q} + \frac{A_2}{ax+b}$$

(iv) For repeated quadratic factor $(lx^2+px+q)^n$

i.e if $g(x) = (lx^2+px+q)^2(ax+b)$

$$\frac{f(x)}{g(x)} = \frac{A_1x+B_1}{lx^2+px+q} + \frac{A_2x+B_2}{(lx^2+px+q)^2} + \frac{A_3}{ax+b}.$$

Problems:-

Q-1: $\int \frac{4x+5}{x^2+x-2} \cdot dx$

$= \int \frac{4x+5}{(x+2)(x-1)} \cdot dx$ ——— (1)

$\frac{4x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$ ——— (2)

$\frac{4x+5}{(x+2)(x-1)} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$

$\Rightarrow \frac{4x+5}{(x+2)(x-1)} = \frac{Ax-A+Bx+2B}{(x+2)(x-1)}$

$\Rightarrow \frac{4x+5}{\cancel{(x+2)(x-1)}} = \frac{(A+B)x-A+2B}{\cancel{(x+2)(x-1)}}$

$\Rightarrow 4x+5 = (A+B)x - A - 2B$

Now comparing coeffecients of x.

$A+B=4$

$-A+2B=5$

$3B=9$

$\Rightarrow B=3$

$A=4-B$

$A=4-3$

$A=1$

Now putting the value of A & B in eqn (2):-

$\frac{4x+5}{(x+2)(x-1)} = \frac{1}{x+2} + \frac{3}{x-1}$

Now Integrating ~~on both sides~~ with respect to x,

$$\Rightarrow \int \frac{4x+5}{(x+2)(x-1)} \cdot dx = \int\left(\frac{1}{x+2} + \frac{3}{x-1}\right) \cdot dx$$

$$= \int \frac{1}{x+2} \cdot dx + \int \frac{3}{x-1} \cdot dx$$

$$= \ln|x+2| + 3\ln|x-1| + C$$

Ex-2 :- $\int \frac{x^2}{(x+1)^2(x-2)} \cdot dx$

$$\frac{x^2}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

$$\Rightarrow \frac{x^2}{\cancel{(x+1)^2(x-2)}} = \frac{A(x+1)(x-2) + B(x-2) + C(x+1)^2}{(x+1)^2(x-2)}$$

$$= \frac{A(x^2-2x+x-2) + B(x-2) + C(x^2+2x+1)}{\cancel{(x+1)^2(x-2)}}$$

$$\Rightarrow x^2 = (A+C)x^2 - (A-B-2C)x - 2C - 2B + C$$

Now comparing the coefficient of x,

$$\Rightarrow B = \frac{-1}{3}$$

$$C = \frac{4}{9}$$

$$A = \frac{5}{9}$$

$$\therefore \frac{x^2}{(x+1)^2(x-2)} = \frac{\frac{5}{9}}{x+1} - \frac{\frac{1}{3}}{(x+1)^2} + \frac{\frac{4}{9}}{x-2}$$

$$= \frac{5}{9} \times \frac{1}{x+1} - \frac{1}{3}\frac{1}{(x+1)^2} + \frac{4}{9}\frac{1}{(x-2)}$$

Now, Integrating on both sides w r to x.

$$\Rightarrow \int \frac{x^2}{(x+1)^2(x-2)}\cdot dx = \frac{5}{9}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{dx}{(x+1)^2} + \frac{4}{9}\int \frac{dx}{x-2}$$

$$= \frac{5}{9}\ln|x+1| - \frac{1}{3}\times\frac{1}{x+1} + \frac{4}{9}\ln|x-2| + C$$

(iii) $\int \frac{2x^2+x+3}{(x^2+2)(x-1)}\cdot dx$

$$\frac{2x^2+x+3}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1}$$

$$\therefore \quad 2x^2+x+3 = (Ax+B)(x-1) + C(x^2+2)$$

Now comparing the coeffecient of x^2, x ..

$A = 0, B = 1, C = 2$

$$\therefore \frac{2x^2+x+3}{(x^2+2)(x-1)} = \frac{1}{x^2+2} + \frac{2}{x-1}$$

Now Integrating on both side with respect to x, we get

$$\int \frac{2x^2+x+3}{(x^2+2)(x-1)}\cdot dx = \int \frac{dx}{x^2+2} + 2\int \frac{dx}{x-1}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + 2\ln|x-1| + C$$

:-

(vii) Integral of the form

$$\int \frac{x^2+1}{x^4+\lambda x^2+1}\,dx,\ \int \frac{x^2-1}{x^4+\lambda x^2+1}\,dx,\ \int \frac{1}{x^4+\lambda x^2+1}\,dx,$$

Algorithm

step-I: Divide numerator & denominator by x^2

step-II: Express the denominator of integrand in the form $\left(x+\frac{1}{x}\right)^2 \pm k^2$

step-III: Introduce $d\left(x+\frac{1}{x}\right)$ or $d\left(x-\frac{1}{x}\right)$ or both in the numerator.

step-IV Substitute $x+\frac{1}{x}=t$ or $x-\frac{1}{x}=t$ as the case may be.
This substitution reduces the integrals in one of the following forms $\int \frac{1}{x^2+a^2}\,dx$, $\int \frac{1}{x^2-a^2}\,dx$.

step-V Use the appropriate formula.

Problems: —

$$\int \frac{x^2-1}{x^4+x^2-1}\,dx$$

$$= \int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx$$

$$= \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-(1)^2}$$

Let $x + \frac{1}{x} = u$

$\Rightarrow d\left(x + \frac{1}{x}\right) = du$

$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = du$

$= \int \frac{du}{(u^2-1)^2} = \frac{1}{2(1)} \log\left|\frac{u-1}{u+1}\right| + C$

$= \frac{1}{2} \log\left|\frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1}\right| + C$

$= \frac{1}{2} \log\left|\frac{x^2 - x + 1}{x^2 + x + 1}\right| + C$

INTEGRATION OF SOME SPECIAL IRRATIONAL ALGEBRAIC FUNCTION :—

(I) Integral of the form $\int \frac{dx}{(ax+b)\sqrt{cx+d}}$ & $\int \frac{1}{(ax^2+bx+c)\sqrt{px+q}}$

Ex: (I) $\int \frac{1}{(x-3)\sqrt{x+1}}\,dx$

let $x+1 = t^2 \Rightarrow dx = 2t\cdot dt$

$$= \int \frac{1}{(t^2-1-3)} \times \frac{2t}{\sqrt{t^2}}\,.dt$$

$$= 2\int \frac{dt}{t^2-2^2} = 2 \times \frac{1}{2(2)} \log\left|\frac{t-2}{t+2}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right| + C$$

(II) $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,.dx$

let $x+1 = t^2$

$\Rightarrow dx = 2t\cdot dt$

$$= \int \frac{(t^2+2)2t}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}\,.dt$$

$$= 2\int \frac{(t^2+1)}{t^4+t^2+1}\,.dt = 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\,.dt$$

$$= 2\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{3})^2}\,dt = 2\int \frac{du}{u^2+(\sqrt{3})^2}, \text{ where } t-\frac{1}{t}=u$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left\{\frac{t - \frac{1}{t}}{\sqrt{3}}\right\} + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{3}\,t}\right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left\{\frac{x}{\sqrt{3(x+1)}}\right\} + C$$

Integral of the form $\int \frac{1}{(ax+b)\sqrt{px^2+qx+r}}\,dx$:

Put $ax+b = \frac{1}{t}$

Ex: $\int \frac{1}{(x+1)\sqrt{x^2-1}}\,dx$

Let $x+1 = \frac{1}{t}$

$\Rightarrow dx = \frac{-1}{t^2}\,dt$

$$= \int \frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2 - 1}} \times \left(\frac{-1}{t^2}\right) dt$$

$$= -\int \frac{dt}{\sqrt{1-2t}} = -\int (1-2t)^{-1/2}\,dt = \frac{-(1-2t)^{1/2}}{(-2)\left(\frac{1}{2}\right)} + C$$

$$= \sqrt{1-2t} + C$$

$$= \sqrt{1 - \frac{2}{x+1}} + C$$

$$= \sqrt{\frac{x-1}{x+1}} + C$$

Integration of the form $\int \frac{1}{(ax^2+b)\sqrt{cx^2+d}}\,dx$:-

Here put $x = \frac{1}{t}$

Ex:- $\int \frac{1}{x^2(\sqrt{1+x^2})}\,dx$

let $x = \frac{1}{t}$ & $\frac{-1}{x^2}\,dx = dt$

$\Rightarrow dx = -x^2\,dt$

$$= \int \frac{-dt}{\sqrt{1+\frac{1}{t^2}}}$$

$$= -\int \frac{t\,dt}{\sqrt{t^2+1}} = -\int \frac{u\,du}{\sqrt{u^2}} \quad \text{, where } t^2+1 = u^2$$

$$= \int -1\,du = -u + c = -\sqrt{t^2+1} + C = -\sqrt{\frac{1}{x^2}+1} + C$$

$$= -\frac{\sqrt{1+x^2}}{x} + C$$

Trigonometric fun^n of the form $\int \frac{dx}{a+b\cos x + c\sin x}$:-

This can be evaluated by converting $\cos x$ & $\sin x$ to $\tan \frac{x}{2}$ $(=t)$

Ex:- $\int \frac{dx}{2+\sin x}$

$$= \int \frac{dx}{2 + \frac{2\tan x/2}{1+\tan^2 x/2}}$$

$$= \frac{1}{2}\int \frac{1+\tan^2 x/2}{\tan^2 x/2 + 1 + \tan x/2}\,dx$$

$$= \frac{1}{2}\int \frac{\sec^2 x/2}{\tan^2 x/2 + \tan x/2 + 1}\,dx$$

$$= \int \frac{dt}{t^2+t+1}, \quad \text{Where } \tan x/2 = t$$

$$= \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan x/2 + 1}{\sqrt{3}}\right) + c$$

DEFINITE INTEGRAL: —

Fundamental Theorem of Integral Calculus: —

Statement: – Let $\phi(x)$ be the primitive or antiderivative of a continuous function $f(x)$ defined on $[a,b]$ i.e $\frac{d}{dx}[\phi(x)] = f(x)$. Then the definite Integral of $f(x)$ over $[a,b]$ is denoted by $\int_a^b f(x).dx$ & is equal to $[\phi(b) - \phi(a)]$

i.e; $\int_a^b f(x).dx = \phi(b) - \phi(a)$.

Where $a \rightarrow$ Lower limit

$b \rightarrow$ Upper limit

$[a,b] \rightarrow$ Interval of Integration

Evaluation of Definite Integral :—

Algorithm :—

Step-I : Find the Indefinite Integral $\int f(x)\cdot dx$

let this be $\phi(x)$.

There is no need to Keep constant of integration.

Step-II :- Evaluate $\phi(b)$ & $\phi(a)$

Step-III :- Calculate $\phi(b) - \phi(a)$ & this will be the answere.

Q(1)- $\displaystyle\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

(2) $\displaystyle\int_0^1 \frac{1}{2x-3}\,dx = \left[\frac{1}{2}\log(2x-3)\right]_0^1$

$$= \frac{1}{3}\left[\log|-1| - \log|-3|\right]$$

$$= \frac{1}{2}\left(\log 1 - \log 3\right)$$

$$= \frac{1}{2}\left(0 - \log 3\right) = -\frac{\log 3}{2}$$

Elementary properties of Definite Integrals :-

(1) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(2) $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(z)\,dz$

(3) $\displaystyle\int_a^b f(x)\,dx = \int_a^\alpha f(x)\,dx + \int_\alpha^b f(x)\,dx$, where $a < \alpha < b$

(4) $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(5) $\displaystyle\int_{-a}^a f(x)\,dx = \begin{cases} 2\displaystyle\int_0^a f(x)\,dx & \text{if } f \text{ is even function} \\ 0 & \text{, if } f \text{ is odd function.} \end{cases}$

(6) $\displaystyle\int_0^{2a} f(x)\,dx = \begin{cases} 2\displaystyle\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x). \end{cases}$

Ex:- (i) $\displaystyle\int_1^4 [x]\,dx = \int_1^2 [x]\,dx + \int_2^3 [x]\,dx + \int_3^4 [x]\,dx$

$\displaystyle = \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 3\,dx$

$= 2-1+2(3-2)+3(4-3) = 6.$

(ii) $\displaystyle\int_{-3}^4 |x|\,dx = \int_{-3}^0 |x|\,dx + \int_0^4 |x|\,dx$

$\displaystyle = \int_{-3}^0 (-x)\,dx + \int_0^4 x\,dx = \int_0^{-3} x\,dx + \int_0^4 x\,dx$

$\displaystyle = \left|\frac{x^2}{2}\right|_0^{-3} + \left|\frac{x^2}{2}\right|_0^4 = \frac{9}{2} - 0 + \frac{16}{2} - 0 = \frac{25}{2}$

(5) $\int_0^{\pi/2} \ln \sin x \cdot dx$

$$I = \int_0^{\pi/2} \ln \sin\left(\frac{\pi}{2} - x\right) \cdot dx = \int_0^{\pi/2} \ln \cos x \cdot dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \ln \cos x \cdot dx + \int_0^{\pi/2} \ln \sin x \cdot dx$$

$$= \int_0^{\pi/2} (\ln \cos x + \ln \sin x)\, dx$$

$$= \int_0^{\pi/2} \ln(\cos x \times \sin x) \cdot dx$$

$$= \int_0^{\pi/2} \ln\left(\frac{\sin 2x}{2}\right) \cdot dx$$

$$2I = \int_0^{\pi/2} \ln \sin 2x \cdot dx - \int_0^{\pi/2} \ln 2 \cdot dx \quad \text{——— (1)}$$

Now $\int_0^{\pi/2} \ln \sin 2x \cdot dx$

$$= \frac{1}{2} \int_0^{\pi} \ln \sin z \, dz, \quad \text{putting } z = 2x$$

$$= \int_0^{\pi/2} \ln \sin x \cdot dx = I$$

Now from eqⁿ (1)

$$\Rightarrow 2I = I - \int_0^{\pi/2} \ln 2 \cdot dx$$

$$= I - \frac{\pi}{2} \ln 2$$

$$\Rightarrow I = -\frac{\pi}{2} \ln 2 = \frac{\pi}{2} \ln\left(\frac{1}{2}\right)$$

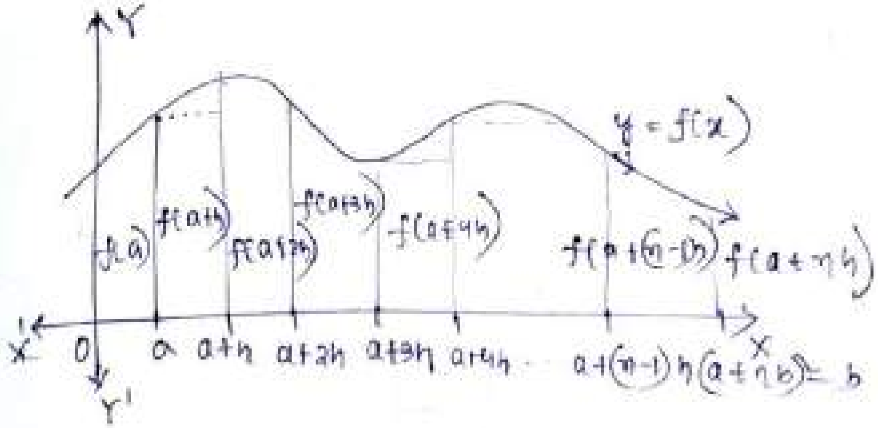
Area under plane curves :—

Integration As the Limit of A Sum :—

Let $f(x)$ be a continuous real valued function defined on the closed interval $[a, b]$ which is divided into n equal parts each of ~~which~~ width h by inserting $(n-1)$ points $a+h, a+2h, a+3h, \dots a+(n-1)h$ between a & b, then

$nh = b-a$

$\Rightarrow h = \dfrac{b-a}{n}$



Let S_n = Sum of Areas of n rectangles

$\Rightarrow S_n = h\cdot f(a) + h f(a+h) + h\cdot f(a+2h) + \dots + h\cdot f(a+(n-1)h)$

$= h[f(a) + f(a+h) + f(a+2h) + \dots f(a+(n-1)h)]$

Now as $n \to \infty$, S_n gives area of the region bounded by the curve $y = f(x)$, $y = 0$ (x-axis) & ordinates $x = a$ & $x = b$.

$\Rightarrow$ This $\lim\limits_{n \to \infty} S_n$ exist for all continuous functions defined on closed integral $[a, b]$ which is the definite integral of $f(x)$ over $[a, b]$.

$\therefore \lim_{n \to \infty} S_n = \int_a^b f(x).dx$

or $\int_a^b f(x).dx = \lim_{n \to \infty} h\left[f(a) + f(a+h) + f(a+2h) + \cdots f(a+(n-1)h)\right]$

where $h = \frac{b-a}{n}$

or $\lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \cdots f(a+(n-1)h)\right]$ $[\because n \to \infty \Leftrightarrow h \to 0]$

→ This method is also called Integration by ab-initio method or integration as the limit of a sum.

Note:-

① $1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}$

② $1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}$

③ $1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 = \left\{\frac{n(n-1)}{2}\right\}^2$

④ $a + ar + ar^2 + \cdots + ar^{n-1} = a\left(\frac{r^n - 1}{r-1}\right), r \neq 1$

Problems:—

(1) $\int_0^2 (x+4)\,dx$

Now we have

$$\int_a^b f(x).\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\cdots f(a+(n-1)h)\right]$$

where $h=\frac{b-a}{n}$

Here $a=0, b=2, f(x)=x+4$ & $h=\frac{2-0}{n}=\frac{2}{n}$

$$\therefore \int_0^2 (x+4)\,dx = \lim_{h\to 0} h\left[f(0)+f(0+h)+f(0+2h)+\cdots f(0+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[(0+4)+(h+4)+(2h+4)+\cdots+((n-1)h+4)\right]$$

$$= \lim_{h\to 0} h\left[4n + h(1+2+3+\cdots+(n-1))\right]$$

$$= \lim_{h\to 0} h\left\{4n + h\,\frac{n(n-1)}{2}\right\}$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[4n+\frac{2}{n}\times\frac{n(n-1)}{2}\right] \qquad \left[h=\frac{2}{n} \;\&\; h\to 0 \Rightarrow n\to\infty\right]$$

$$= \lim_{n\to\infty}\left\{8+2\left(1-\frac{1}{n}\right)\right\} = 8+2(1-0)=10$$

(2) $\int_0^2 e^x.\,dx$

We have, $\int_a^b f(x).\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h)\right]$

where $h=\frac{b-a}{n}$

Here $a = 0,\ b = 2,\ f(x) = e^x,\ h = \frac{2}{n}$

$$\int_0^2 e^x \cdot dx = \lim_{h \to 0} h\left[f(0) + f(h) + f(2h) + \cdots f((n-1)h)\right]$$

$$= \lim_{h \to 0} h\left[e^0 + e^h + e^{2h} + \cdots e^{(n-1)h}\right]$$

$$= \lim_{h \to 0} h\left[e^0 \left\{\frac{(e^h)^n - 1}{e^h - 1}\right\}\right]$$

$$= \lim_{h \to 0} h\left\{\frac{e^{nh} - 1}{e^h - 1}\right\}$$

$$= \lim_{h \to 0} \frac{h}{h}\left\{\frac{e^2 - 1}{\left(\frac{e^h - 1}{h}\right)}\right\} \qquad \left[h = \frac{2}{n} \Rightarrow nh = 2\right]$$

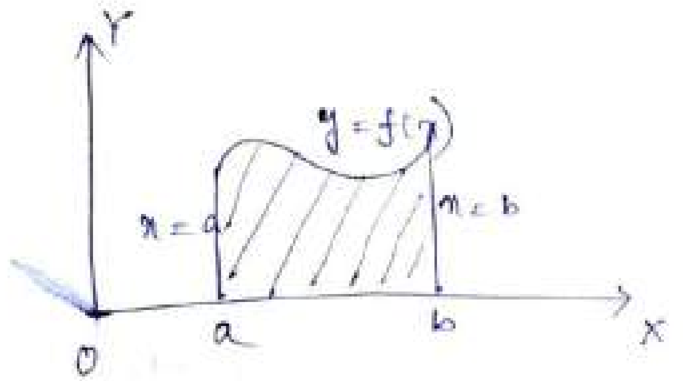
$$= \lim_{h \to 0} \frac{e^2 - 1}{\left(\frac{e^h - 1}{h}\right)} = \frac{e^2 - 1}{1} = e^2 - 1 \qquad \left[\lim_{h \to 0} \frac{e^h - 1}{h} = 1\right]$$

Areas As a Definite Integral: —

Theorem: — Let $f(x)$ be a continuous function defined on $[a, b]$. Then the area bounded by the curve $y = f(x)$, the x-axis & the ordinates $x = a$ & $x = b$ is given by $\int_a^b f(x) \cdot dx$ or $\int_a^b y \cdot dx$.

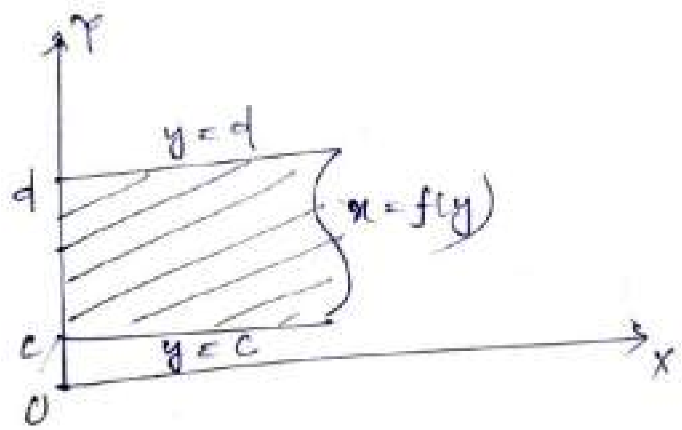
Def.1 :- Area of the region bounded by the curve $y = f(x)$ the X-axis & the lines $x = a$, $x = b$ is defined by

$$\text{Area} = \left|\int_a^b y \cdot dx\right| = \left|\int_a^b f(x) \cdot dx\right|$$



Def^n. 2 :- Area of the region by the curve $x = f(y)$, the Y-axis & the lines $y = c$, $y = d$ is defined by

$$\text{Area} = \int_c^d x \cdot dy = \left|\int_c^d f(y) \cdot dy\right|$$



Problems :-

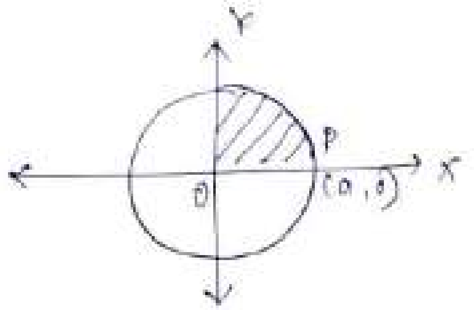
① Find the area of the region bounded by the curve $y = e^{3x}$, X-axis & the lines $x = 4$ & $x = 2$.

Ans: The required area is defined by.

$$A = \int_2^4 e^{3x}\,dx = \frac{1}{3}e^{3x}\Big]_2^4 = \frac{1}{3}\left(e^{12x} - e^{6x}\right)$$

Ex-2: Find the area of the circle $x^2+y^2=a^2$?

Solⁿ: We observe that, $y=\sqrt{a^2-x^2}$ in the first quadrant



∴ The area of the circle in the first quadrant is defined by,

$$A_1 = \int_0^a \sqrt{a^2-x^2}\,dx$$

$$\text{Total Area } (A) = 4\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

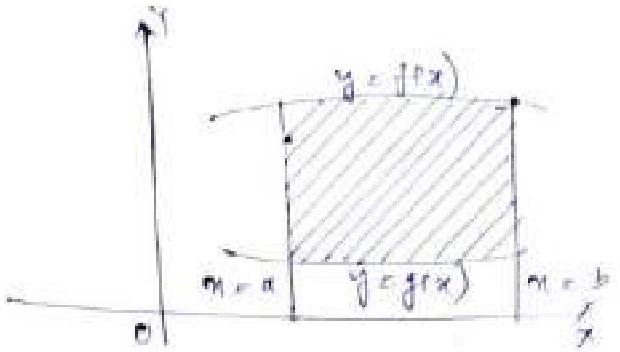
$$= 4\frac{a^2}{2}\sin^{-1}1 = 2a^2\frac{\pi}{2} = \pi a^2.$$

Area Between Two curves: —

If there two curves $y=f(x)$, $y=g(x)$ with $g(x) < f(x)$ in $[a, b]$, then the area between them & between the ordinates $x=a$ & $x=b$ is given by

$$A = \int_a^b f(x)\cdot dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\cdot dx$$

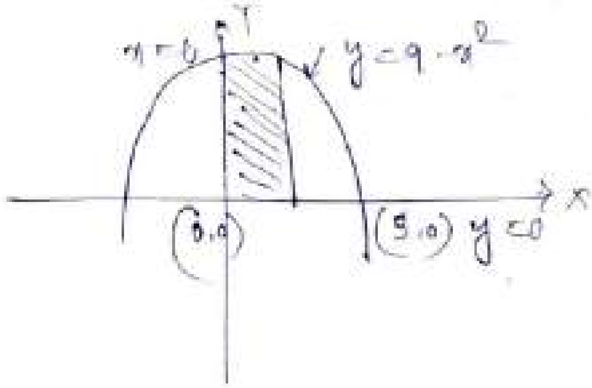


Ex 1.?(i) Area of the region enclosed by $y = 9 - x^2$, $y = 0$ & the ordinates $x = 0$ & $x = 2$ is given by

$$A = \int_0^2 (9 - x^2)\,dx$$

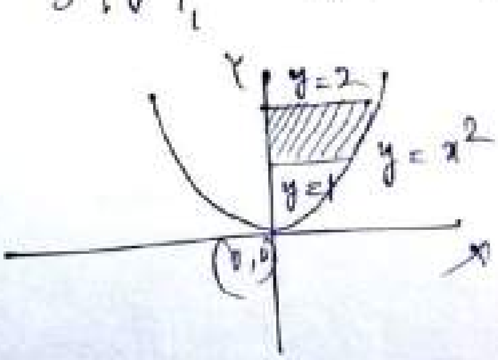
$$= \left[9x - \frac{x^3}{3}\right|_0^2$$

$$= 18 - \frac{8}{3} = \frac{46}{3}$$



(ii) The area bounded by $x^2 = y$, the y-axis $(x = 0)$ & the lines $y = 1$ & $y = 2$ is given by

$$A = \int_1^2 x\cdot dy = \int_1^2 \sqrt{y}\cdot dy = \frac{2}{3}\left|y^{\frac{3}{2}}\right|_1^2 = \frac{2}{3}\left(2\sqrt{2} - 1\right)$$



Assignments →

(1) Find the Integrals :-

(a) $\int 4x^3 \cdot dx$

(b) $\int x^6 \cdot dx$

(c) $\int \left(2\sqrt{x} + \frac{3}{\sqrt{x}}\right) dx$

(d) $\int \left(x^{1/3} + \frac{1}{x^{1/3}}\right) dx$

(e) $\int \frac{1 - \cos 2x}{1 + \cos 2x} \cdot dx$

(f) $\int \frac{a\sin^3 x + b\cos^3 x}{\sin^2 2x} \cdot dx$

(g) $\int 3^x \cdot dx$

(h) $\int \left(\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}\right) dx$

Topic No. 05

Topic Name : Differential Equation.

Contents to be covered :

(a) Introduction to differential eqn

(b) Order & degree of differential eqn

(c) Solution of differential eqn

(i) By separable Methd

(ii) Linear differential eqn

1.1 Definition of differential eqn

An equation relating an unknown function and one or more of its derivatives is called a differential eqn.

∴ The differential equation $\frac{dy}{dx} = x + \sin x$ involves both unknown functions $y(x)$ and $y'(x) = \frac{dy}{dx}$

∴ The differential equation

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} \cdot \frac{dy}{dx} + y = x$$ involves unknown functions y and x and first three derivatives y''', y'' & y'

1.2 Types of differential eqn

(a) Ordinary differential eqn (ODE)

(b) Partial differential eqn (PDE)

Defn (a) A differential eqn involving unknown functions (dependent variable) depends only on single independent variable $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + ty = x$ is ODE. here y depends on x as independent variable.

Defn (b) If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; if they are the eqn $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ is called PDE.

1.3 Order of a differential equation

It is the order of highest derivative which appears in the differential eq^n

Ex $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^5 = e^x$ is a differential equation of fourth order.

1.4 Degree of differential equation

It is the power of highest derivative that occurs in a differential equation after the differential equations has been made free from radicals and fractions as far as the derivative is concerned.

NOTE If any term of differential equation cannot be expressed as a polynomial in the derivatives, then the degree of differential eq^n is not defined.

Ex $\left(\frac{d^2y}{dx^2}\right)^2 - \sin\left(\frac{dy}{dx}\right) = 0$

Here order = 2, Degree is not defined.

Ex find order and degree of the following differential eq^n. $y = x\cdot\frac{dy}{dx} + a\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$

Sol^n We are given $y = x\cdot\frac{dy}{dx} + a\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$

$y - x\frac{dy}{dx} = a\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$

Squaring on both sides

$\left(y - x\frac{dy}{dx}\right)^2 = \left[a\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}\right]^2$

$$\Rightarrow y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} = a^2 + a^2\left(\frac{dy}{dx}\right)^2$$

Now the above equation is free from all radicals, so here

Order = 1

Degree = 2

Assignment:

Write down the order and degree of the following differential equations.

(i) $\dfrac{d^2x}{dy^2} + \sqrt{\left[1+\left(\dfrac{dx}{dy}\right)^3\right]} = 0$

(ii) $\dfrac{d^2x}{dy^2} + \sqrt{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]} = 0$

(iii) $\left(\dfrac{d^3y}{dx^3}\right)^2 - xy\left(\dfrac{dy}{dx}\right)^3 + y = 0$

(iv) $\left(\dfrac{dy}{dx}\right) = \sin x$

(v) $\left(\dfrac{d^2y}{dx^2}\right)^3 - xy\left(\dfrac{dy}{dx}\right)^4 + y = 0$

(vi) $\left[1+\left\{\dfrac{dy}{dx}\right\}^2\right]^{\frac{3}{2}} = e^x$

(vii) $\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) = \sin x$

(viii) $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^3 + a^2\left(\dfrac{d^2y}{dx^2}\right)^2 = 0$

(ix) $\left(\dfrac{d^2y}{dx^2}\right)^{\frac{3}{2}} - \left(\dfrac{dy}{dx}\right)^{\frac{1}{2}} - 4 = 0$

(x) $\left(\dfrac{d^2y}{dx^2}\right)^3 + 4 \cdot \dfrac{\left(\dfrac{d^2y}{dx^2}\right)^3}{\dfrac{d^3y}{dx^3}} + \dfrac{d^3y}{dx^3} = x^2 - 1$

Solution of differential eq^n

A solution of a differential eq^n is any relation between the variables involved which satisfies the differential equation.

Here we are going to find the solution of differential eq^n of first order and first degree.

2.1 Introduction

There are two standard forms of differential equation of first order and first degree.

i.e (i) $\frac{dy}{dx} = f(x,y)$ (ii) $M(x,y)dx + N(x,y)dy = 0$

We now discuss various methods to solve such equations.

2.2 Separation of variables

A first order differential equation

$\frac{dy}{dx} = H(x,y)$ is called separable

provided $H(x,y)$ can be written as the product of function of 'x' and function of 'y'

$$\frac{dy}{dx} = g(x) \cdot h(y) = \frac{g(x)}{f(y)}, \text{ let } h(y) = \frac{1}{f(y)}$$

Now $f(y)\,dy = g(x)\,dx$

Now integrating both sides

$$\int f(y)\,dy = \int g(x)\,dx + C.$$

Some problems based on separation of variables

Example-01 Solve $\frac{dy}{dx} = e^{x-y}$

Soln.: $\frac{dy}{dx} = e^{x-y}$

$\Rightarrow \frac{dy}{dx} = e^{x} \cdot e^{-y}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{x}}{e^{y}}$

$\Rightarrow e^{y} dy = e^{x} dx$

Now integrating on both sides.

$\Rightarrow \int e^{y} dy = \int e^{x} dx + C$

$\Rightarrow e^{y} = e^{x} + c$ Ans

Example-02. Solve $(e^{x}+1) y\, dy = (y+1) e^{x} dx$

Soln Separating the variables

$(e^{x}+1) y\, dy = (y+1) e^{x} dx$

$\Rightarrow \frac{y}{y+1} dy = \frac{e^{x}}{e^{x}+1} dx$

Now integrating on both sides.

$\Rightarrow \int \frac{y}{y+1} dy = \int \frac{e^{x}}{e^{x}+1} dx + c$

$\Rightarrow \int \left(1 - \frac{1}{y+1}\right) dy = \log(e^{x}+1) + c$

$\Rightarrow y - \log(y+1) = \log(e^{x}+1) + c$

Ans

Example-03 Solve $x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0$

Sol^n: Separating the variables

$$x\sqrt{1+y^2}\,dx = -y\sqrt{1+x^2}\,dy$$

$$\Rightarrow \frac{x}{\sqrt{1+x^2}}\,dx = -\frac{y}{\sqrt{1+y^2}}\,dy$$

Now integrating on both side,

$$\int\frac{x}{\sqrt{1+x^2}}\,dx = -\int\frac{y}{\sqrt{1+y^2}}\,dy + c$$

$$\Rightarrow \sqrt{1+x^2} = -\sqrt{1+y^2} + c$$

$$\Rightarrow \sqrt{1+x^2} + \sqrt{1+y^2} = c \quad \text{Ans}$$

Example-04 $3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$

Sol^n Separating the variables.

$$\frac{3e^x}{1-e^x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$$

Integrating on both sides.

$$3\int\frac{e^x}{1-e^x}\,dx + \int\frac{\sec^2 y}{\tan y}\,dy = \log c$$

$$\Rightarrow -3\log(1-e^x) + \log(\tan y) = \log c$$

$$\Rightarrow \log\frac{1}{(1-e^x)^3} + \log(\tan y) = \log c$$

$$\Rightarrow \log\frac{\tan y}{(1-e^x)^3} = \log c$$

$$\Rightarrow \frac{\tan y}{(1-e^x)^3} = c \Rightarrow \tan y = (1-e^x)^3 c.$$

Linear Differential equation.

A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{--- (1)}$$

On an interval on which the co-efficients $P(x)$ and $Q(x)$ are contineous, is called linear differential equation.

We multiply both sides of eq (1) by the Integrating factor $\text{I.F.} = e^{\int P(x)dx}$

The result is

$$e^{\int P(x)dx} \cdot \frac{dy}{dx} + P(x) \cdot e^{\int P(x)dx} \cdot y = Q(x) \cdot e^{\int P(x)dx}$$

$$\frac{d}{dx}\left[y(x)\, e^{\int P(x)dx}\right] = Q(x)\, e^{\int P(x)dx}$$

Now integrating on both side,

$$y(x) \cdot e^{\int P(x)dx} = \int \left(Q(x) \cdot e^{\int P(x)dx}\right)dx + C$$

So finally solving for y, we obtain the general Sol^n of first order linear equation

$$y(x) = e^{-\int P(x)dx}\left[\left(\int Q(x) \cdot e^{\int P(x)dx}\right)dx + c\right]$$

If $\frac{dx}{dy} + P'(y)\,x = Q'(y)$, where P' and Q' are constant or functions of y, then $IF = e^{\int P'(y)dy}$.

Example-01 Solve $(1+y^2) + (x - e^{-\tan^{-1}y})\frac{dy}{dx} = 0$

Sol^n: We can rewrite the given eq^n as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{-\tan^{-1}y}}{1+y^2}$$

Which is the linear differential eq^n of the form $\frac{dx}{dy} + P'(y)x = Q'(y)$

Then $IF = e^{\int P'(y)dy} = e^{\int \frac{1}{1+y^2}dy}$

$$= e^{\tan^{-1}y}$$

Solution is given by

$$x \cdot IF = \int Q(y) \cdot IF \, dy + C$$

$$\Rightarrow x \cdot e^{\tan^{-1}y} = \int \frac{e^{-\tan^{-1}y}}{1+y^2} \cdot e^{\tan^{-1}y} dy + C$$

$$\Rightarrow x e^{\tan^{-1}y} = \int \frac{1}{1+y^2} dy + C$$

$$\Rightarrow x e^{\tan^{-1}y} = \tan^{-1}y + C \quad \text{Ans}$$

Example-02 Solve $x\cos x\left(\frac{dy}{dx}\right) + y(x\sin x + \cos x) = 1$

Sol^n: We rewrite the given eq^n as

$$\frac{dy}{dx} + \frac{y(x\sin x + \cos x)}{x\cos x} = \frac{1}{x\cos x}$$

$$\Rightarrow \frac{dy}{dx} + y\left(\tan x + \frac{1}{x}\right) = \frac{\sec x}{x}.$$

$$I \cdot F = e^{\int \left(\tan x + \frac{1}{x}\right)dx} = e^{\log \sec x + \log x}$$

$$= e^{\log x\sec x} = x\sec x.$$

Now Solution will be

$$y \cdot IF = \int Q(x) \cdot IF \, dx + C$$

$$\Rightarrow y \cdot x\sec x = \int \frac{\sec x}{x} \cdot x\sec x \, dx + C$$

$$\Rightarrow y \cdot x\sec x = \int \sec^2 x \, dx + C$$

$$\Rightarrow y \cdot x\sec x = \tan x + C \quad \text{Ans}$$

Example 09   Solve $(x+1)\frac{dy}{dx} - y = e^x (x+1)^2$

Sol^n. We can rewrite the given eq^n as

$$\frac{dy}{dx} - \frac{1}{x+1} y = \frac{e^x (x+1)^2}{x+1}$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{1+x} y = e^x (1+x)$$

$$IF = e^{\int \frac{-1}{1+x} dx} = e^{-\log(x+1)} = e^{\log\left(\frac{1}{x+1}\right)}$$

$$= \frac{1}{x+1}$$

Sol^n will be

$$y \cdot IF = \int Q(x) \cdot IF \, dx + C$$

$$\Rightarrow y \cdot \frac{1}{x+1} = \int e^x (x+1) \frac{1}{x+1} dx + C$$

$$\Rightarrow y \cdot \frac{1}{x+1} = \int e^x \, dx + C$$

$$\Rightarrow y \cdot \frac{1}{x+1} = e^x + C \quad \text{Ans}$$

Example 09 $\frac{dy}{dx} + \frac{y}{x} = x \cdot y^2$

Solution: Dividing both sides by y^2

We get

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\cdot\frac{1}{x} = x$$

let put $\frac{1}{y} = z$

$$-\frac{1}{y^2}\cdot\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow -\frac{dz}{dx} + \frac{z}{x} = x$$

$$\Rightarrow \frac{dz}{dx} - \frac{1}{x}\cdot z = -x$$

Which is a linear differential eq^n in z

Now $I \cdot F = e^{\int -\frac{1}{x} dx}$

$$= e^{-\log x}$$

$$= e^{\log \frac{1}{x}} = \frac{1}{x}$$

Solution will be

$$z \cdot IF = \int Q(x)\, I\cdot F\, dx + C$$

$$\Rightarrow z \cdot \frac{1}{x} = \int (-x) \cdot \frac{1}{x}\, dx + C$$

$$\Rightarrow \frac{z}{x} = -\int dx + C$$

$$\Rightarrow \frac{z}{x} = -x + C$$

$$\Rightarrow z = x(c - x)$$

$$\Rightarrow \frac{1}{y} = x(c - x)$$

$$\Rightarrow y = \frac{1}{x(c-x)}$$ Ans.

Assignments.

Solve the following differential eqn

1. $\frac{dy}{dx} + \frac{y}{x} = x^2$ if $y=1$ when $x=1$

2. $x^2\left(\frac{dy}{dx}\right) + y = 1$

3. $y\,dx - x\,dy + \log x\,dx = 0$

4. $\cos x\left(\frac{dy}{dx}\right) + y = \sin x$

5. $(1-x)\,dy + (1-y)\,dx = 0$

6. $x\,dy + 2y\,dx = xy\,dy$

7. $\tan y\,dx + \tan x\,dy = 0$

8. $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0$

9. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

10. $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

11. $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$

12. $\sin x \frac{dy}{dx} + 3y = \cos x$

13. $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$

14. $2(1+y^2)\,dx + (2x - \tan^{-1}y)\,dy = 0$

15. $\frac{dy}{dx} = (x^2+1)(y^2+1)$

16. $(x^2-1)\frac{dy}{dx} + 2xy = 1$

17. $\frac{dy}{dx} = x\cos x$

18. $\frac{dy}{dx} = \frac{e^{2x}+1}{e^x}$

19. $\frac{dy}{dx} = \sec y$

20. $\frac{dy}{dx} = y^2 + 2y$