A. ALGEBRA

MATRICES AND DETERMINANT

MATRICES:

[History of the Matrix: The matrix has a long history of application in solving linear equations. They were known as arrays until the 1800‘s. The term “matrix” (Latin for “womb”, derived from *mater*—mother) was coined by James Joseph Sylvester in 1850, who understood a matrix as an object giving rise to a number of determinants today called minors, that is to say, determinants of smaller matrices that are derived from the original one by removing columns and rows. An English mathematician named Cullis was the first to use modern bracket notation for matrices in 1913and he simultaneously demonstrated the first significant use of the notation A=(ai,j)to represent a matrix where ai,j refers to the element found in the *i*th row and the *j*th column. Matrices can be used to compactly write and work with multiple linear equations, referred to as a system of linear equations.]

Definition

Matrix (whose plural is matrices) is a rectangular array of numbers (or other mathematical objects), arranged in rows and columns, for which operations such as addition and multiplication are defined. The numbers are called the elements, or entries, of the matrix. Generally the capital letters of the alphabets are used to denote matrices and the matrices are commonly written in box brackets or parentheses ([ ] , ( ))

Example:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

If there are m rows and n columns in a matrix, it is called a ” m by n” matrix or a matrix of order m×n, where m is the number of rows and n is the number of columns..

Example:

$$A= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$$

A is a matrix of order 2×3 (matrix with two rows and three columns)

$$B= \begin{bmatrix} 5 & 2 \\ 6 & 1 \\ 7 & 3 \end{bmatrix}$$

B is a matrix of order 3×2 (matrix with three rows and two columns)

Types of matrices:

1. **Row matrix:** Matrix with a single row is called a row matrix

$$A=\begin{bmatrix} a & b & c \end{bmatrix}$$

A is a row matrix (1×3) with one row and 3 columns

$$B=[a_{11} \ldots \ldots \ldots . . a_{1n}]$$

B is also a row matrix (1×n) with 1 row and n columns.

2. **Column matrix:** Matrix with a single column is called a column matrix

$$A=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

A is a column matrix (3×1) with 3 rows and 1 column

$$B=\begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{n1} \end{bmatrix}$$

B is column matrix($n\times1$) with n rows and 1 column.

3. **Null matrix:** a matrix is said to be a null matrix or zero matrix if all its entries are zero it is noted by $O_{m\times n,}$ if it has m rows and n columns.

Example:

$$O_{2\times3}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4. **Square matrix:** if the number of rows and columns of a matrix are equal the it is said to be a square matrix..

Example:

$$A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

A is a square matrix(3x3) of order 3 where number of rows and columns are each 3.

5. **Diagonal matrix :** a square matrix of which the non -diagonal elements are all zero is called a diagonal matrix.

Example:

$$A=\begin{bmatrix} a_{11} & 0 \dots \dots 0 \\ 0 & a_{22} \dots \dots 0 \\ . & \\ . & \\ 0 & 0 \dots \dots a_{nn} \end{bmatrix}$$

A is a diagonal matrix of order n

Example:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$ is a diagonal matrix of order 3

6. **Scalar matrix:** if the diagonal elements of a diagonal matrix are all equal it is called a scalar matrix.

Example:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$ is a scalar matrix of order 3

7. **Identity matrix:** if the diagonal elements of a diagonal matrix are all unity (1) , it is called a unit matrix.

Example:

$$I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is a unit matrix of order 3

A unit matrix is also called identity matrix. A unit matrix of order n is denoted by I_n or I.

8. **Transpose of a matrix**: Transpose of a matrix is obtained just by changing its rows into columns and columns into rows. It is denoted by A^T OR A^I

Example:

If $A=\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, then A^T or A^I = $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$

(A is (2x3) matrix whereas A^T is (3x2) matrix)

If $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then $A^T=\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$ (both are (3x3) matrices)

Algebra of matrices

a) Equality of matrices: Two matrices A and B are said to be equal if and only if

i. The order of A is equal to that of B

ii. Each element of A is equal to the corresponding element of B.

Example:

$[x \quad y]=[1 \quad 2]$ ⇒x= 1 and y= 2

But, $\begin{bmatrix} x \\ y \end{bmatrix} \neq [1 \quad 2]$

As the order of $\begin{bmatrix} x \\ y \end{bmatrix}$ is 2×1 where as order of $[1 \quad 2]$ is 1×2.

b) Addition of matrices: The sum of two matrices A and B is the matrix such that each of its elements is equal to the sum of the corresponding elements of A and B. The sum is denoted by A+B .Thus the addition of matrices is defined if they are of same order and is not defined when they are of different orders.

Example:

If $A=\begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 5 \\ 0 & -2 & 1 \end{bmatrix}$ $B=\begin{bmatrix} 2 & 3 & 1 \\ 0 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

A+B is defined as the order of A and B are same (3×3)

$$A+B=\begin{bmatrix} 1+2 & 2+3 & 0+1 \\ 3+0 & 1-2 & 5+2 \\ 0+1 & -2+2 & 1-1 \end{bmatrix}=\begin{bmatrix} 3 & 5 & 1 \\ 3 & -1 & 7 \\ 1 & 0 & 0 \end{bmatrix}$$

Which is also of order (3×3)

If $A=\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 1 & 3 \\ 2 & 2 & 1 \end{bmatrix}$ and $C=\begin{bmatrix} 4 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

Then $A+B+C=\begin{bmatrix} 2+2+4 & 3+1+3 & 4+3+1 \\ 1+2+2 & 2+2+3 & 3+1+4 \end{bmatrix}=\begin{bmatrix} 8 & 7 & 8 \\ 5 & 7 & 8 \end{bmatrix}$

Three matrices of order(2×3) are added and the sum is a matrix of the order (2 ×3) .

If $A=[1 \quad 2]$, $B=\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

Then A+B is not defined as the order of A and B are not same

Properties:

1. The addition of matrices is **commutative**

If A and B are two matrices of same order, then A+B = B+A

Proof:

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices of same order

Then, $A + B = (a_{ij} + b_{ij}) = (b_{ij} + a_{ij}) = B + A$

Example:

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$

Then

$$A + B = \begin{bmatrix} 1+2 & 2+1 \\ 3+2 & 4+2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 5 & 6 \end{bmatrix}$$

$$B + A = \begin{bmatrix} 2+1 & 1+2 \\ 2+3 & 2+4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 5 & 6 \end{bmatrix}$$

Hence $A + B = B + A$

2. The matrix addition is **associative**

If A,B and C are three matrices of same order, then $A + (B + C) = (A + B) + C$

Proof:

Let $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$ be three matrices of same order.

Then,

$$A + (B + C) = (a_{ij} + (b_{ij} + c_{ij})) = ((a_{ij} + b_{ij}) + c_{ij}) = (A + B) + C$$

Example:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 \\ 3 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 1+(3+2) & 3+(2+1) \\ 2+(1+3) & 2+(1+3) \\ 3+(2+1) & 1+(3+2) \end{bmatrix}$$

$$= \begin{bmatrix} (1+3)+2 & (3+2)+1 \\ (2+1)+3 & (2+1)+3 \\ (3+2)+1 & (1+3)+2 \end{bmatrix} = (A + B) + C$$

3. Zero matrix (O) is the **identity matrix** for addition

$$A + O = A$$

Proof:

Let $A = (a_{ij})$, Then, $A + O = (a_{ij} + 0) = (a_{ij}) = A$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A + O = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = A$$

4. Additive inverse of a matrix

The matrix in which each element is the negative of the corresponding element of a given matrix A, is called the negative of A and is denoted by (–A). The matrix –A is called the **additive inverse** of the matrix A.

i.e If $A=(a_{ij})$, Then $–A=(-a_{ij})$

and $A+(-A)=(a_{ij}+(-a_{ij}))=(0)=O$

Example:

If $A=\begin{bmatrix}2 & 3\\5 & -6\end{bmatrix}$ then, $-A=\begin{bmatrix}-2 & -3\\-5 & 6\end{bmatrix}$

and $A+(–A)= \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

i.e $A+(-A)=O$

Again if A+B=O. Then A is the additive inverse of B and B is the additive inverse of A.

c) Subtraction

The subtraction of two matrices A and B of the same order is defined by

$$A–B = A+(–B)$$

Example:

If $A = \begin{bmatrix}1 & 2 & -1\\2 & 1 & 0\\4 & -3 & -2\end{bmatrix}$, $B=\begin{bmatrix}2 & 1 & -3\\0 & 5 & -1\\3 & 4 & -2\end{bmatrix}$

Then $A–B = A+(–B)$

$$= \begin{bmatrix}1 & 2 & -1\\2 & 1 & 0\\4 & -3 & -2\end{bmatrix} + \begin{bmatrix}-2 & -1 & 3\\0 & -5 & 1\\-3 & -4 & 2\end{bmatrix} = \begin{bmatrix}-1 & 1 & 2\\2 & -4 & 1\\1 & -7 & 0\end{bmatrix}$$

d) Product of a matrix and a scalar

The product of a scalar m and a matrix A , denoted by mA is the matrix whose elements is m times the corresponding elements of A . Thus if ,

If $A=(a_{ij})$, Then $mA=(ma_{ij})$

Example:

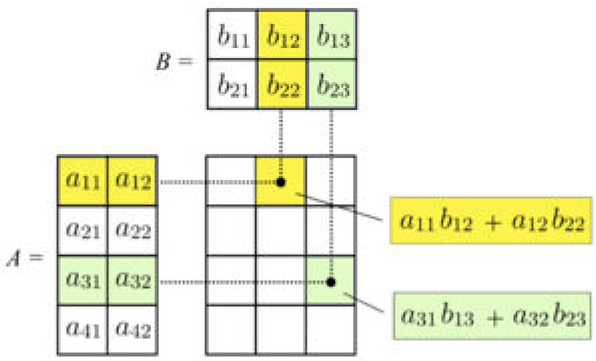
$A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$, then $mA =\begin{bmatrix}ma & mb\\mc & md\end{bmatrix}$

Example:

If $A= \begin{bmatrix}2 & 1\\3 & -2\end{bmatrix}$ then, $2A=\begin{bmatrix}2\times 2 & 2\times 1\\2\times 3 & 2\times -2\end{bmatrix} =\begin{bmatrix}4 & 2\\6 & -4\end{bmatrix}$

e) Matrix multiplication

Multiplication of two matrices is defined if and only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If A is an (*m*xp) matrix and B is an (pxn) matrix, then their matrix product AB is the (mxn) matrix whose entries are given by product of the corresponding row of A and the corresponding column of B as shown



(Schematic depiction of the matrix product AB of two matrices A and B.)

The product of two matrices A and B (where the number of columns in A=the number of rows in B) is the matrix AB whose elements in the ith row and jth column is the sum of the products formed by multiplying each element in the ith row of A by the corresponding element in the jth column of B.

If A= (a_{ij}) is a (mxp) matrix
and B= (b_{ij}) is a (pxn) matrix
Then, AB=(c_{ij}) is a (mxn) matrix, where $c_{ij}=\sum_{k=1}^{k=p} a_{ik}b_{kj}$

Example:

If A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3\times3}$

The product, AB is defined because number of column of A = number of rows of B.
In this case the order of AB is (2×3)
We have,

$$AB=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

Applying $c_{ij}=\sum_{k=1}^{k=3} a_{ik}b_{kj}$, we get,

$$=\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \end{bmatrix}$$

Example:

If A=$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & -1 \end{bmatrix}_{2\times3}$ and B=$\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$

Then the product AB is defined and given as

$$AB= \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$=\begin{bmatrix} (1\times-1)+(2\times2)+(3\times3) \\ (-2\times-1)+(1\times2)+(-1\times3) \end{bmatrix}$$

$=\begin{bmatrix} -1+4+9 \\ 2+2-3 \end{bmatrix}=\begin{bmatrix} 12 \\ 1 \end{bmatrix}_{2\times 1}$

Properties:

1. The multiplication of matrices is **Not always commutative** that is if A and B are any two matrices then AB≠BA

case-1: If A and B are matrices of different orders such that the product AB is defined, but BA is not defined or if BA is defined but AB is not defined.

Example:

If $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, then AB defined, but BA not defined,

Therefore, AB≠BA

case-2: If A and B are square matrices of same order, then the product AB and BA are both defined.

But AB≠BA, in general

Example-1:

If $A=\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ Then

$AB=\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 4 & 3 \\ -1 & 3 \end{bmatrix}$

$BA=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 3 & 3 \\ -1 & 4 \end{bmatrix}$

$\therefore$ AB ≠ BA

But in some cases matrix multiplication is commutative i.e. AB=BA

Example-2

If $A=\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ (scalar matrix) and $B=\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Then we have

$AB=\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix}$

$BA=\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}=\begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix}$

Hence AB =BA

Example-3: if $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (unit matrix). Then

$AB=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$BA=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Hence AB =BA

2. The multiplication of matrix is **associative**

If A,B,C are three matrices such that the products (AB)C and A(BC)are defined,

Then (AB)C= A(BC)

Example: Let A=$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix}$, B=$\begin{bmatrix} 1 & 5 \\ 0 & -3 \\ 2 & 1 \end{bmatrix}$ and C=$\begin{bmatrix} 0 & 2 & -1 \\ -3 & 4 & 2 \end{bmatrix}$.Then

AB=$\begin{bmatrix} 1+0+6 & 5-6+3 \\ 4+0+2 & 20+0+1 \end{bmatrix}$=$\begin{bmatrix} 7 & 2 \\ 6 & 21 \end{bmatrix}$

(AB)C=$\begin{bmatrix} 7 & 2 \\ 6 & 21 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 \\ -3 & 4 & 2 \end{bmatrix}$=$\begin{bmatrix} 0-6 & 14+8 & -7+4 \\ 0-63 & 12+84 & -6+42 \end{bmatrix}$=$\begin{bmatrix} -6 & 22 & -3 \\ -63 & 96 & 36 \end{bmatrix}$

BC=$\begin{bmatrix} 1 & 5 \\ 0 & -3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 \\ -3 & 4 & 2 \end{bmatrix}$=$\begin{bmatrix} 0-15 & 2+20 & -1+10 \\ 0+9 & 0-12 & 0-6 \\ 0-3 & 4+4 & -2+2 \end{bmatrix}$=$\begin{bmatrix} -15 & 22 & 9 \\ 9 & -12 & -6 \\ -3 & 8 & 0 \end{bmatrix}$

A(BC)=$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} -15 & 22 & 9 \\ 9 & -12 & -6 \\ -3 & 8 & 0 \end{bmatrix}$=$\begin{bmatrix} -15+18-9 & 22-24+24 & 9-12+0 \\ -60+0-3 & 88+0+8 & 36+0+0 \end{bmatrix}$

=$\begin{bmatrix} -6 & 22 & -3 \\ -63 & 96 & 36 \end{bmatrix}$

Therefore, (AB)C= A(BC)

3. **Identity matrix of multiplication**

The identity matrix of multiplication for the set of all square matrices of a given order is the unit matrix of the same order.

Example-1:

If A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and I=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then we have,

AI= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ =$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

IA=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

Therefore, AI = IA = A

Example-2:

If A= $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and I=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then we have

AI= $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$=$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ = A

IA=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ = $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ = A

i.e AI = IA = A

DETERMINANT

To every square matrix A of order n, we can associate a number (real or complex) called determinant of the matrix A, written as det A or $|A|$. In the case of a 2×2 matrix the determinant may be defined as

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then, $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

Notes:

i. Only square matrices have determinants.
ii. For a matrix A, $|A|$ is read as determinant of A and not, as modulus of A.

Determinant is used in the solution of linear algebraic equations.

Consider the two equations,

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

Solving this system of equations, we get

$$x=(b_1c_2 - b_2c_1)/(a_1b_2 - a_2b_1)$$

and $y = (c_1a_2 - c_2a_1)/(a_1b_2 - a_2b_1)$

This solution exists, provided $a_1b_2-a_2b_1 \neq 0$

The quantity $(a_1b_2-a_2b_1)$ determines whether a solution of the linear equation exists or not and is denoted by the symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, which is called a determinant of order 2.

Thus $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2-a_2b_1$

The determinant is also sometimes denoted by the symbol Δ .

Similarly, the system of equation

$$a_1x + b_1y + c_1z + d_1 = 0$$

$$a_2x + b_2y + c_2z + d_2 = 0$$

$$a_3x + b_3y + c_3z + d_3 = 0$$

admits a solution if, $(a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_2 - a_2b_1c_3 + a_2b_3c_1 - a_3b_2c_1) \neq 0$

The above expression can be denoted by

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, which is called a determinant of order 3

i.e $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_2 - a_2b_1c_3 + a_2b_3c_1 - a_3b_2c_1$

Minor and cofactor

Minor: Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting ith row and j th column, and it is denoted by M_{ij}.

In a determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Minor of a_{11} = $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ = M_{11}

Minor of a_{12} = $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ = M_{12} and so on.

Cofactor: The cofactor of an element a_{ij} is defined as $(-1)^{i+j}M_{ij}$ where M_{ij} is the minor of a_{ij} . It is denoted by C_{ij}

i.e C_{ij} = cofactor of a_{ij} $=(-1)^{i+j}M_{ij}$

$C_{11} = (-1)^{1+1}M_{11} = M_{11}$

$C_{12}=(-1)^{1+2}M_{12} = -M_{12}$

$C_{13} =(-1)^{1+3}M_{13} = M_{13}$ and so on..

Expansion of determinants

Example:

(a) For determinants order 2

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

(b) For higher order determinants:

A determinant is evaluated by expanding the determinant by the elements of any row (or any column) as the sum of products of the elements of the row (column)with the cofactors of the respective elements of the same row (column). Thus There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) and each way gives the same value.

$\Delta=\begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix}=a_1(-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} +b_1(-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}+(-1)^{1+3}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$= a_1(b_2c_3-b_3c_2) - b_1(a_2c_3-c_2a_3)+c_1(a_2b_3- b_2a_3)$

$=a_1C_{11} + b_1C_{12} + c_1C_{13}$

(where C_{ij} is the cofactor of the element corresponding to ith row jth column)

The above expansion has been made using the elements of the 1st row

Example:

Let $\Delta=\begin{vmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \\ 4 & -1 & 2 \end{vmatrix}=2\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}-3\begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix}+4\begin{vmatrix} -1 & 2 \\ 4 & -1 \end{vmatrix}$

= 2(4+3) -3(-2-12) +4(1-8)

= 2(7) -3(-14) + 4(-7)

=14 +42 -28 =56-28 =28

Properties of determinant:

Property 1: The value of the determinant is not altered by changing the rows into columns and the columns into rows.

i.e $|A'| = |A|$, where A' = transpose of matrix A.

Example:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

L.H.S = $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

R.H.S= $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

Example:

$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$, as

$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10-12=-2$

$\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 10-12=-2$

Property 2: If two adjacent rows or columns of a determinant are interchanged then the sign of the determinant changes without changing its numerical value.

Example:

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$ (changing 1^{st} and 2^{nd} row)

Or $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$ (changing 1^{st} and 2^{nd} column)

As,

$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

$\Delta_1 = \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = a_2b_1 - a_1b_2 = -(a_1b_2 - a_2b_1) = -\Delta$

Example:

$\Delta = \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = 2-12 = -10$

$\Delta_1 = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 12-2 = 10 = -\Delta$

Property 3: if two rows or two columns of a determinant are identical then the value of the determinant is zero.

Example: (Two rows are equal)

$\begin{vmatrix} a_1 & b_1 \\ a_1 & b_1 \end{vmatrix} = a_1b_1 - a_1b_1 = 0$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_2 & c_2 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_2 & c_2 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_2 & b_2 \end{vmatrix}$$

$$= a_1(b_2c_2 - b_2c_2) - b_1(a_2c_2 - a_2c_2) + (a_2b_2 - a_2b_2) = 0$$

Example: (Two columns are equal)

$$\begin{vmatrix} a_1 & a_1 \\ b_1 & b_1 \end{vmatrix} = a_1b_1 - a_1b_1 = 0$$

$$\begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_2 \\ b_3 & b_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + b_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2b_3 - b_2b_3) - b_1(a_2b_3 - a_3b_2) + b_1(a_2b_3 - a_3b_2) = 0$$

Property 4: If each elements of any row or any column is multiplied by the same factor then the determinant is multiplied by that factor

$$\begin{vmatrix} ma & mb \\ c & d \end{vmatrix} = m \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

As, $\begin{vmatrix} ma & mb \\ c & d \end{vmatrix} = mad - mbc = m(ad - bc) \quad = m \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

As,

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = ma_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - mb_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + mc_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= m\left\{a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right\}$$

$$= m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example :

$$\begin{vmatrix} 50 & 100 & 150 \\ 4 & 5 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 50 \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\text{L.H.S} = \begin{vmatrix} 50 & 100 & 150 \\ 4 & 5 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 50 \begin{vmatrix} 5 & 1 \\ 1 & 3 \end{vmatrix} - 100 \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} + 150 \begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix}$$

$$= 50(15 - 1) - 100(12 - 2) + 150(4 - 10)$$

$$= (50 \times 14) - (100 \times 10) + (150 \times -6)$$

$$= 700 - 1000 - 900 = -1200$$

$$\text{R.H.S} = 50 \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 50 \left[1 \begin{vmatrix} 5 & 1 \\ 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix}\right]$$

$$= 50\{14 - 20 - 18\} = 50 \times -24 = -1200$$

Note:

If any two rows or any two columns in a determinant are proportional, then the value of the determinant is also zero.

Property 5 : If elements of a row or a column in a determinant can be expressed as the sum of two or more elements, then the given determinant can be expressed as the sum of two or more determinants of the same order.

Example:

$$\begin{vmatrix} a_1+\alpha_1 & b_1 \\ a_2+\alpha_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 \\ \alpha_2 & b_2 \end{vmatrix}$$

$$\text{L.H.S} = \begin{vmatrix} a_1+\alpha_1 & b_1 \\ a_2+\alpha_2 & b_2 \end{vmatrix} = (a_1+\alpha_1)b_2 - (a_2+\alpha_2)b_1$$

$$= a_1b_2 + \alpha_1 b_2 - a_2b_1 - \alpha_2 b_1$$

$$= (a_1b_2 - a_2b_1) + (\alpha_1 b_2 - \alpha_2 b_1)$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 \\ \alpha_2 & b_2 \end{vmatrix} = \text{R.H.S}$$

Property 6: If to each element of a row or a column of a determinant the equimultiples of corresponding elements of other rows (columns) are added, then value of determinant remains same.

Example:

$$\begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\text{L.H.S} = \begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} kc & kd \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + k\begin{vmatrix} c & d \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + k \times 0 = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \text{R.H.S}$$

Notes:

I. If all the elements of a row (or column) are zeros, then the value of the determinant is zero.

II. If value of determinant ‘Δ’ becomes zero by substituting $x = \alpha$, then $x - \alpha$ is a factor of ‘Δ’.

III. If all the elements of a determinant above or below the main diagonal are zeros, then the value of the determinant is equal to the product of diagonal elements.

Adjoint of a Matrix :

If A is a square matrix, then the transpose of the matrix of which the elements are cofactors of the corresponding elements on A is called the adjoint of A and denoted by Adj A.

Example:

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

cof $A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$ where c_{ij} is the cofactor corresponding to the element a_{ij}.

Then Adj A= $(\text{cof } A)^T$ (Transpose of the cofactor matrix)

$$= \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

Example 1:

Let A= $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

Here C_{11} =1, C_{12}= –3, C_{21}=–2, C_{22}=1

Cof A =$\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$

Adj A= (Cof A)^T = $\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}^T$ =$\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$

Example 2:

Let A=$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

Here

C_{11}= $\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$ = 1-2 = -1, C_{12} = - $\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$ = -(2-4) = 2, C_{13} = $\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}$ = 0,

C_{21}= -$\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$ = -(2-1) = -1, C_{22}= $\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$ =1-2 = -1, C_{23}= - $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ = -(1-4) = 3,

C_{31}= $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$ = 4-1 = 3, C_{32}= - $\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$ = 0, C_{33}= $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ = 1-4 = -3

Cof A =$\begin{bmatrix} -1 & 2 & 0 \\ -1 & -1 & 3 \\ 3 & 0 & -3 \end{bmatrix}$

Adj A= (CofA)^T =$\begin{bmatrix} -1 & 2 & 0 \\ -1 & -1 & 3 \\ 3 & 0 & -3 \end{bmatrix}^T$ =$\begin{bmatrix} -1 & -1 & 3 \\ 2 & -1 & 0 \\ 0 & 3 & -3 \end{bmatrix}$

Theorem-1

If A is a square matrix then A.(AdjA)=$|A|$ I =(Adj A). A

Proof;-

Let A= $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

Adj A=$\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$

A.(Adj A) = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}c_{11} + a_{12}c_{12} & a_{11}c_{21} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{12} & a_{21}c_{21} + a_{22}c_{22} \end{bmatrix}$$

$$=\begin{bmatrix} |A| & 0 \\ 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|\, I$$

Hence A.(Adj A) = |A| I

If |A| =0 then A.(Adj A) is zero matrix. In this case the matrix A is said to be a singular matrix.

i.e Matrix A is singular if |A| =0 and non-singular if $|A| \neq 0$.

Inverse of a Matrix:

If A and B are two square matrices of the same order such that AB=BA=I

Then B is called the multiplicative inverse of A.

B is written as A^{-1} or B= A^{-1}

Also, A is called the inverse of B and is written as B^{-1} or A= B^{-1}

If A is a non-singular matrix, then A^{-1} exists and the inverse is given by

$$A^{-1}= \frac{1}{|A|}(\text{ Adj A})$$

Proof:

From theorem-1, we have

A. (Adj A) = |A| I

Or, $A\left(\frac{AdjA}{|A|}\right)$ =I

Therefore, $A^{-1}=\frac{1}{|A|}$ Adj A

Example1

$$A=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$|A|= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2-1=1 \neq 0$$

A is a non-singular matrix. Hence A^{-1} exists.
We know that

$$A^{-1} = \frac{1}{|A|}(\text{ Adj A})$$

Here $C_{11} = 1,$ $C_{12} = -1$

$C_{21} = -1,$ $C_{22} = 2$

Cof A $=\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$,

Adj A= (Cof A)^T $=\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{ Adj A})=\frac{1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

Example 2:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$|A|= \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1\text{-}6= \text{-}5 \neq 0$$

Hence A^{-1} exists.

Here $C_{11} = 1$, $C_{12} = -3$

$C_{21} = -2,$ $C_{22} = 1$

$$\text{Adj } A = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

We know that

$$A^{-1} = \frac{1}{|A|}(\text{ Adj A}) = \frac{1}{-5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{-1}{5} \end{bmatrix}$$

Example 3

Let, $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$. Then

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} -2\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} +1\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$$=1(1-6) -1(2-2)+1(6-1) = -5-0+5 = 0$$

Here $|A| = 0$.
i.e A is a singular matrix.
Hence, inverse of A does not exists.

Example 4

Let, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Then, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} -2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} +3\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$

$$=1(2-1) – 2(4-1)+ 3 (2-1)$$

$$=1-6+3= -2 \neq 0$$

Hence A^{-1} exists

Now

$$C_{11} = 1, \quad C_{12} = -3, \quad C_{13} = 1$$

$$C_{21} = -1, \quad C_{22} = -1, \quad C_{23} = 1$$

$$C_{31} = -1, \quad C_{32} = 5, \quad C_{33} = -3$$

$$\text{Adj A} = (\text{cofA})^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -3 & -1 & 5 \\ 1 & 1 & -3 \end{bmatrix}$$

We know that,

$$A^{-1} = \frac{1}{|A|}(\text{ Adj A})$$

$$= \frac{1}{-2}\begin{bmatrix} 1 & -1 & -1 \\ -3 & -1 & 5 \\ 1 & 1 & -3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{-5}{2} \\ \frac{-1}{2} & \frac{-1}{2} & \frac{3}{2} \end{bmatrix}$$

Cramer's Rule:-

Cramer's rule is used in the solution of simultaneous linear equations

Consider the equations in two variables

$$a_1x + b_1y = d_1$$

$$a_2x + b_2y = d_2$$

Solving these two equations by using cross multiplication method we have

$$\frac{x}{d_1b_2-d_2b_1} = \frac{y}{a_1d_2-a_2d_1} = \frac{1}{a_1b_2-a_2b_1}$$

i.e $\frac{x}{D_x} = \frac{y}{D_y} = \frac{1}{D}$

Where,

$$D = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

$$D_x = d_1b_2 - d_2b_1 = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}, \quad D_y = a_1d_2 - a_2d_1 = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$$

Therefore, $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$

Consider the equations in three variables

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Here,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Now multiplying D by x, we have

$$xD = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} \quad (C_1 \to C_1 + yC_2 + zC_3)$$

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = D_x$$

Or $xD = D_x$

Or, $\frac{x}{D_x} = \frac{1}{D}$

Similarly, we can show that $\frac{y}{D_y} = \frac{1}{D}$ and $\frac{z}{D_z} = \frac{1}{D}$

Where,

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad , \quad D_Z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Therefore, $x = \frac{D_X}{D}, \; y = \frac{D_Y}{D}, \; z = \frac{D_Z}{D}$

Example 1

Consider, $x + 2y = 5$

$3x + y = 7$

Here D= $\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$ = 1-6 = -5 ≠ 0

The system admits a solution

$$D_x = \begin{vmatrix} 5 & 2 \\ 7 & 1 \end{vmatrix} = 5 - 14 = -7$$

$$D_y = \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = 7 - 15 = -8$$

By Cramer's rule,

$$x = \frac{D_x}{D} = \frac{-7}{-5} = \frac{7}{5} \;,\; y = \frac{D_y}{D} = \frac{-8}{-5} = \frac{8}{5}$$

Example 2

Consider, $x + y = 3$

$2x + 2y = 7$

Here D= $\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0$

The system does not possess a solution.

Example 3

Consider, $x - 2y + z = 2$

$6x - 9y + z = 1$

$-9x + 12y + z = 4$

Here, D= $\begin{vmatrix} 1 & -2 & 1 \\ 6 & -9 & 1 \\ -9 & 12 & 1 \end{vmatrix}$

$=1\begin{vmatrix} -9 & 1 \\ 12 & 1 \end{vmatrix} + 2\begin{vmatrix} 6 & 1 \\ -9 & 1 \end{vmatrix} + 1\begin{vmatrix} 6 & -9 \\ -9 & 12 \end{vmatrix}$

$= -21 + 30 - 9 = 0$

Similarly, $D_X = 0 \;, D_Y = 0 \;, D_Z = 0$

So, the system has infinite number of solution.

Example 4

Consider, $x + 2y + 3z = 6$

$2x + 4y + z = 7$

$3x + 2y + 9z = 1$

Here $D=\begin{vmatrix}1 & 2 & 3\\2 & 4 & 1\\3 & 2 & 9\end{vmatrix}$

$=1\begin{vmatrix}4 & 1\\2 & 9\end{vmatrix}-2\begin{vmatrix}2 & 1\\3 & 9\end{vmatrix}+3\begin{vmatrix}2 & 4\\3 & 2\end{vmatrix}$

$=1(36-2)-2(18-3)+3(4-12)$

$= 34 – 32 – 24$

$= -20 \neq 0$

The system of equations admits a solution

Similarly, we have,

$$D_x = \begin{vmatrix}6 & 2 & 3\\7 & 4 & 1\\14 & 2 & 9\end{vmatrix} = -20$$

$$D_y = \begin{vmatrix}1 & 6 & 3\\2 & 7 & 1\\3 & 14 & 9\end{vmatrix} = -20$$

$$D_z = \begin{vmatrix}1 & 2 & 6\\2 & 4 & 7\\3 & 2 & 14\end{vmatrix} = -20$$

By Cramer's rule.

$$x=\frac{D_X}{D}=\frac{-20}{-20}=1,\quad y=\frac{D_Y}{D}=\frac{-20}{-20}=1,\quad z=\frac{D_Z}{D}=\frac{-20}{-20}=1$$

Solution of simultaneous linear equations by matrix inverse method

Let us consider two linear equations with two variables

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

The above system of equations can be written as

$AX = B$

Where $A=\begin{bmatrix}a_1 & b_1\\a_2 & b_2\end{bmatrix}$, $X=\begin{bmatrix}x\\y\end{bmatrix}$ and $B=\begin{bmatrix}c_1\\c_2\end{bmatrix}$

Or, $X=A^{-1}B$

$=\frac{1}{|A|}(\mathrm{Adj}A)\,B,$

if $|A| \neq 0$ Then the system admits solution.

Similarly, for three linear equation with three variables

$a_{11}x + a_{12}y + a_{13}z = b_1$

$a_{21}x + a_{22}y + a_{23}z = b_2$

$a_{31}x + a_{32}y + a_{33}z = b_3$

The above system of equations can be written in matrix form as

$AX = B$, where A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B= $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Then,

$X = A^{-1}B$

Or, $X = \frac{1}{|A|}(Adj\ A)B,$

If $|A| \neq 0$ Then the system admits solution

Example 1

Consider the following system of linear equations

3x-4y = 1

2x + y = 8

The above system can be written as

AX=B

Where A = $\begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} 1 \\ 8 \end{bmatrix}$

Or, $X = A^{-1}B = \frac{1}{|A|}(\mathrm{AdjA})\ \mathrm{B}$

Here, $|A|$=$\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix}$ =3-(-8)= 11 ≠ 0

So, the system of equations admits solution.

Here, $C_{11} = 1,\ \ C_{12} = -2$

$C_{21} = 4,\ \ C_{22} = 3$

Cof A = $\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$

Adj A= $(\text{CofA})^T$ =$\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$

So that, we have

$$\mathrm{X} = \frac{1}{|A|}(\mathrm{AdjA})\mathrm{B} = \frac{1}{11}\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 1+32 \\ -2+24 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 33 \\ 22 \end{bmatrix} = \begin{bmatrix} 33/11 \\ 22/11 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Hence, $x = 3$ and y = 2

Example 2

Let us consider the system of equations,

$x – y + z = 2$

$2x + y – 3z = 5$

$3x – 2y – z = 4$

The system of equations can be written in matrix form as

$$AX = B, \quad \text{where } A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 3 & -2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

Or, $X = A^{-1}B = \frac{1}{|A|}(\mathrm{Adj}A)B$

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 3 & -2 & -1 \end{vmatrix} = 1(-1-6) — 1(-2+9) + 1(-4-3) = -7+7-7 = -7 \neq 0$$

So, A^{-1} exist and the system admits solution.

We have,

$C_{11} = -7, \quad C_{12} = -7, \quad C_{13} = -7$

$C_{21} = -3, \quad C_{22} = -4, \quad C_{23} = -1$

$C_{31} = 2, \quad C_{32} = 5, \quad C_{33} = 3$

So that,

$$\text{Cof } A = \begin{bmatrix} -7 & -7 & -7 \\ -3 & -4 & -1 \\ 2 & 5 & 3 \end{bmatrix}$$

$$\text{Adj } A = (\text{Cof}A)^T = \begin{bmatrix} -7 & -7 & -7 \\ -3 & -4 & -1 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} -7 & -3 & 2 \\ -7 & -4 & 5 \\ -7 & -1 & 3 \end{bmatrix}$$

So that, we have,

$$X = \frac{1}{|A|}(\mathrm{Adj}A)\, B = (1/-7)\begin{bmatrix} -7 & -3 & 2 \\ -7 & -4 & 5 \\ -7 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

$$= (1/-7)\begin{bmatrix} -14-15+8 \\ -14-20+20 \\ -14-5+12 \end{bmatrix} = (1/-7)\begin{bmatrix} -21 \\ -14 \\ -7 \end{bmatrix} = \begin{bmatrix} -21/-7 \\ -14/-7 \\ -7/-7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Hence, $x = 3$, $y = 2$, $z = 1$

Some Solved Problems

Q-1: Find the minors and cofactors of all the elements of the matrices

(i) $\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 2 \end{vmatrix}$

Sol:

(i) Given $\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix}$

Let M_{ij} and $C_{ij} = (-1)^{i+j}M_{ij}$ are the minor and cofactors of the element a_{ij}, then

$M_{11} = 4$, $C_{11}=(-1)^{1+1}M_{11}=4$

$M_{12} = -1$, $C_{12}=(-1)^{1+2}M_{12}=1$

$M_{21}=3$, $C_{21}=(-1)^{2+1}M_{21}=-3$

$M_{22}=2$, $C_{22}=(-1)^{2+2}M_{22}=2$

(ii) Given $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 2 \end{vmatrix}$

Let M_{ij} and $C_{ij} = (-1)^{i+j} M_{ij}$ are the minor and cofactors of the element a_{ij}.Then,

$M_{11} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 2 - 12 = -10,$ $C_{11}=(-1)^{1+1}M_{11}= -10$

$M_{12} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1,$ $C_{12}=(-1)^{1+2}M_{12}= -1$

$M_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8 - 1 = 7,$ $C_{13}=(-1)^{1+3}M_{13}= 7$

$M_{21} = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 4 - 4 = 0,$ $C_{21}=(-1)^{2+1}M_{21}= 0$

$M_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1,$ $C_{22}=(-1)^{2+2}M_{22}= 1$

$M_{23} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4 - 2 = 2,$ $C_{23}=(-1)^{2+3}M_{23}= -2$

$M_{31} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5,$ $C_{31}=(-1)^{3+1}M_{31}= 5$

$M_{32} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1,$ $C_{32}=(-1)^{3+2}M_{32}= -1$

$M_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3,$ $C_{33}=(-1)^{3+3}M_{33}= -3$

Q- 2: Evaluate the value of the determinant $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$ using properties

Sol:

$$\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

$$= \begin{vmatrix} 265-240 & 240-219 & 219 \\ 240-225 & 225-198 & 198 \\ 219-198 & 198-181 & 181 \end{vmatrix} \quad (C_1 \leftarrow C_1 - C_2 and C_2 \leftarrow C_2 - C_3)$$

$$= \begin{vmatrix} 25 & 21 & 219 \\ 15 & 27 & 198 \\ 21 & 17 & 181 \end{vmatrix}$$

$$= \begin{vmatrix} 25-21 & 21 & 219-210 \\ 15-27 & 27 & 198-270 \\ 21-17 & 17 & 181-170 \end{vmatrix} \quad (C_1 \leftarrow C_1 - C_2 and C_3 \leftarrow C_3 - 10C_2)$$

$$= \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix}$$

$$= \begin{vmatrix} 4-4 & 21-17 & 9-11 \\ -12+12 & 27+51 & -72+33 \\ 4 & 17 & 11 \end{vmatrix} \quad (\mathrm{R}_1 \to \mathrm{R}_1 - \mathrm{R}_3,\ \mathrm{R}_2 \to \mathrm{R}_2 + 3\mathrm{R}_3)$$

$$= \begin{vmatrix} 0 & 4 & -2 \\ 0 & 78 & -39 \\ 4 & 17 & 11 \end{vmatrix}$$

Expanding with respect to 1st column

$$= 4\begin{vmatrix} 4 & -2 \\ 78 & -39 \end{vmatrix}$$

$$= 4(-156+156) = 4 \times 0 = 0$$

Q- 3: Solve, $\begin{vmatrix} x & a & a \\ m & m & m \\ b & x & b \end{vmatrix} = 0$

Sol:

Given, $\begin{vmatrix} x & a & a \\ m & m & m \\ b & x & b \end{vmatrix} = 0$

Or, $\mathrm{m}\begin{vmatrix} x & a & a \\ 1 & 1 & 1 \\ b & x & b \end{vmatrix} = 0$ (since $m \neq 0$)

Or, $\begin{vmatrix} x-a & 0 & a \\ 0 & 0 & 1 \\ b-x & x-b & b \end{vmatrix} = 0 \quad C_1 \leftarrow C_1 - C_2 and C_2 \leftarrow C_2 - C_3)$

Or, $(-1)\begin{vmatrix} x-a & 0 \\ b-x & x-b \end{vmatrix} = 0$

Or, $(x-a)(x-b) = 0$

Or, $x = a$ or b

Q- 4: Expand $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$, by using properties of determinant

Solution:

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = \begin{vmatrix} 1-1 & 1-1 & 1 \\ x-y & y-z & z \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix} (C_1 \leftarrow C_1 - C_2 and C_2 \leftarrow C_2 - C_3)$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x-y & y-z \\ x^3-y^3 & y^3-z^3 \end{vmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix} 1 & 1 \\ x^2+xy+y^2 & y^2+yz+z^2 \end{vmatrix}$$

$$= (x-y)(y-z)(y^2+yz+z^2-x^2-xy-y^2)$$

$$= (x-y)(y-z)(z-x)(x+y+z)$$

Q- 5 : Factorize $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$

Solution:

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix}$$

$$=\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix} \text{ (Taking xyz common factor from } R_3\text{)}$$

($R_1 \leftrightarrow R_2$, $R_2 \leftrightarrow R_3$)

$$=\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ x^2-y^2 & y^2-z^2 & z^2 \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix} (C_1 \leftarrow C_1 - C_2 \text{and } C_2 \leftarrow C_2 - C_3)$$

$$= \begin{vmatrix} x^2-y^2 & y^2-z^2 \\ x^3-y^3 & y^3-z^3 \end{vmatrix}$$

$$=\begin{vmatrix} (x-y)(x+y) & (y-z)(y+z) \\ (x-y)(x^2+xy+y^2) & (y-z)(y^2+yz+z^2) \end{vmatrix}$$

$$= (x-y)\,(y-z)\begin{vmatrix} x+y & y+z \\ (x^2+xy+y^2) & (y^2+yz+z^2) \end{vmatrix}$$

$$= (x-y)\,(y-z)\begin{vmatrix} x-z & y+z \\ x^2+xy-yz-z^2 & y^2+yz+z^2 \end{vmatrix} \quad (C_1 \leftarrow C_1 - C_2)$$

$$=(x-y)\,(y-z)\begin{vmatrix} x-z & y+z \\ (x-z)(x+y+z) & y^2+yz+z^2 \end{vmatrix}$$

$$= (x-y)\,(y-z)\,(z-x)\begin{vmatrix} 1 & y+z \\ x+y+z & y^2+yz+z^2 \end{vmatrix}$$

Now by expanding we get

$$= (x-y)\,(y-z)(z-x)(xy+yz+zx)$$

Q- 6: Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Proof:

$$\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$=(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -b-c-a & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$$

$$(C_1 \leftarrow C_1 - C_2 and C_2 \leftarrow C_2 - C_3)$$

$$=(a+b+c)\begin{vmatrix} a+b+c & -b-c-a \\ 0 & a+b+c \end{vmatrix}$$

$=(a+b+c)(a+b+c)^2$

$=(a+b+c)^3$

Q- 7 Solve $\begin{vmatrix} 2x+1 & 3 \\ x & 2 \end{vmatrix} = 5$

Solution:

Given $\begin{vmatrix} 2x+1 & 3 \\ x & 2 \end{vmatrix} = 5$

Or, 2(2x+1)-3x = 5

Or, 4x+2 -3x = 5

Or, x+2 = 5

Or, x = 3

Q- 8: Verify that $[AB]^T = B^TA^T$ where $A=\begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 6 & -3 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 5 & 6 & 1 \end{bmatrix}$

Sol:

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 6 & -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 5 & 6 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1+6+15 & 2+8+18 & 3+4+3 \\ 6+21+40 & 12+28+48 & 18+14+8 \\ 6-9+20 & 12-12+24 & 18-6+4 \end{bmatrix} = \begin{bmatrix} 22 & 28 & 10 \\ 67 & 88 & 40 \\ 17 & 24 & 16 \end{bmatrix}$$

$$[AB]^T = \begin{bmatrix} 22 & 67 & 17 \\ 28 & 88 & 24 \\ 10 & 40 & 16 \end{bmatrix}$$

Again, $A^T=\begin{bmatrix} 1 & 6 & 6 \\ 2 & 7 & -3 \\ 3 & 8 & 4 \end{bmatrix}$ and $B^T=\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 2 & 1 \end{bmatrix}$

$$B^TA^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 6 & 6 \\ 2 & 7 & -3 \\ 3 & 8 & 4 \end{bmatrix}$$

$$=\begin{bmatrix} 1+6+15 & 6+21+40 & 6-9+20 \\ 2+8+18 & 12+28+48 & 12-12+24 \\ 3+4+3 & 18+14+8 & 18-6+4 \end{bmatrix} = \begin{bmatrix} 22 & 67 & 17 \\ 28 & 88 & 24 \\ 10 & 40 & 16 \end{bmatrix}$$

Hence, $[AB]^T = B^TA^T$

Q- 9: Write down the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, if a_{ij} = 2i+3j

Sol:

$$\begin{bmatrix} 2+3 & 2+6 & 2+9 \\ 4+3 & 4+6 & 4+9 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \end{bmatrix}$$

Q-10: Construct a 2×3 matrix having elements a_{ij} = i+j

Sol:

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

EXERCISE

1. 02 Marks Questions

I. Evaluate $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$.

II. Solve $\begin{vmatrix} a & b & c \\ b & a & b \\ x & b & c \end{vmatrix} = 0$.

III. Find the minor and cofactor of the elements 4 and 0 in the determinant $\begin{vmatrix} 1 & 2 & -3 \\ 4 & 5 & 0 \\ 2 & -1 & 1 \end{vmatrix}$.

IV. Evaluate $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$.

V. What is the maximum value of $\begin{vmatrix} \sin x & \cos x \\ -\cos x & 1+sinx \end{vmatrix}$.

VI. Without expanding evaluate $\begin{vmatrix} sin^2\theta & cos^2\theta & 1 \\ cos^2\theta & sin^2\theta & 1 \\ -10 & 12 & 2 \end{vmatrix}$,

VII. Without expanding, find the value of $\begin{vmatrix} 1/a & 1 & bc \\ 1/b & 1 & ca \\ 1/c & 1 & ab \end{vmatrix}$.

VIII. If X +$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, Then find X.

IX. Find x and y, if $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

2. 05 Marks Questions

I. Solve $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+x \end{vmatrix} = 0$

II. Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

III. Prove that $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$.

IV. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

V. Prove that $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

VI. Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

VII. Prove that $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$

VIII. Prove that $(AB)^T = B^T A^T$, where $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ -1 & -2 \end{bmatrix}$.

IX. Find the adjoint of the matrix $\begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$

X. If $= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 1 \\ 3 & 1 & -2 \end{bmatrix}$, find adjoint of A.

XI. Find the inverse of the matrices $\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}$

XII. Solve by Cramer's rule: $2x - 3y = 8\,,\quad 3x + y = 1$

XIII. Solve by matrix method : $5x - 3y = 1,\quad 3x + 2y = 12$

3. 10 Marks Questions

I. Prove that $\begin{vmatrix} sin^2 A & \cot A & 1 \\ sin^2 B & \cot B & 1 \\ sin^2 C & \cot C & 1 \end{vmatrix} = 0$, where A, B and C are the angles of a triangle.

II. Prove that $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} = 0$, where $A + B + C = \pi$.

A. TRIGONOMETRY

Introduction

The word 'trigonometry' is derived from two Greek words 'trigonon' and 'metron'. Trigonon means a triangle and metron means a measure. Hence, trigonometry means measurement of triangles, i.e. study of triangles, measurement of their sides, angles and different relations which exist between triangles relates. Initially, this concept was developed to solve geometric problems involving triangles. In earlier days, these ideas were used by sea captains for navigations, surveyor to map the lands, architects/ engineers to construct the buildings, dams, etc and others. But now a days, it's application has extended to many areas like satellite navigations, seismology, measurement of height of a building or mountain, in video games, construction and architecture, flight engineering, cartography (creating maps),in oceanography to measure height of the tides and many other areas also.

The following three different systems of units are used in the measurement of trigonometrical angles

Measurement of an angle:

There are three systems of measurement of an angle.

(i) Sexagesimal system
(ii) Centesimal system
(iii) Circular system

(i) Sexagesimal system

(i) $1\text{right angle} = 90 \text{ degrees}(90°)$
(ii) $1° = 60 \text{ sexagesimal minutes or}(60')$
(iii) $1 \text{ minute or } 1' = 60 \text{ sexagesimal seconds or } 60''$

(ii) Centesimal system

(i) $1 \text{ right angle} = 100\text{grades or } 100^{g}$
(ii) $1^{g} = 100 \text{ centesimal minutes}$
(iii) $1\text{right angle} = 90° = 100^{g}$

(iii) Circular system

The unit of measurement of angles in this system is a radian. A radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of that circle and is denoted by 1^{c}.

$$\frac{\text{circumference}}{\text{diameter}} = \pi,$$

π is a Greek letter, pronounced by "pi" .Two right angles =$180° = 200^{g} = \pi^{c}$

$$1 \text{ radian} = \frac{2}{\pi} \text{ right angle.}$$

NOTE :(i) Angle subtended by an arc length l is $\theta = \frac{l}{r}$

(ii) The angle subtended at the centre of a circle in radians is 2π radians

Trigonometric Ratios

Let XOX’ and YOY’ be two axes of co-ordinates i.e. x-axis and y-axis respectively. These two axes intersect perpendicularly at the point ‘O’, named as Origin.

In x-axis, OX and OX’ are known as positive and negative X-axis respectively. Similarly, In y-axis, OY and OY’ are known as positive and negative Y-axis respectively. Now, both the axes divide the XY-plane into four equal parts called‘quadrants’.

(i) XOY is called 1^{st} quadrant.
(ii) X’OY is called 2^{nd} quadrant..
(iii) X’OY’ is called 3^{rd} quadrant.
(iv) XOY’ is called 4^{th} quadrant.

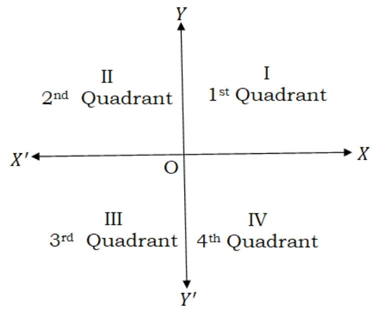


Fig. 2.1

Let us take a point P(x, y). Draw $PM \perp OX$
In Fig 2.2, join OP.
S0 that, $OM = x$, $PM = y$
OPM is a right angle triangle.
If θ is an angle measure such that $0 < \theta < \pi/2$.
Let $\angle POM = \theta$.
So, the side OP opposite of the right angle $\angle PMO$ is known as hypotenuse (h).
The side OM related to right angle and given angle(θ) is base(b) and the side PM is known as the perpendicular(p).

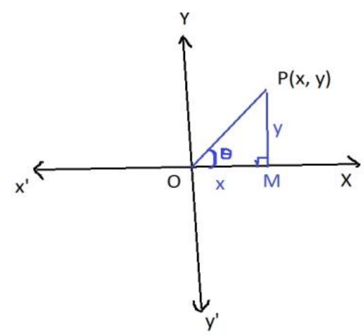


Fig 2.2

Now, in the right angle triangle ΔOPM, (Fig 2.2)
The ratio of its sides (with proper sign) are defined as trigonometrical ratios (T-Ratios).
There are six trigonometric ratios such as sine, cosine, tangent, cotangent, secant, cosecant for θ, abbreviated as sinθ, cosθ, tanθ, cot θ, secθ, cosecθ have been defined as follows.

1. The ratio of the perpendicular to the hypotenuse, is called “sine of the angle θ” and it is written as $\sin\theta$.
   i.e. $\sin\theta = \frac{Perpendicular}{\text{hypotenuse}} = \frac{PM}{OP}$
2. The ratio of the base to the hypotenuse, is called “cosine of the angle θ” and it is written as $\cos\theta$.
   i.e. $\cos\theta = \frac{Base}{\text{Hypotenuse}} = \frac{PM}{OP}$
3. The ratio of the perpendicular to the base, is called “tangent of the angle θ” and it is written as $\tan\theta$.
   i.e. $\tan\theta = \frac{Perpendicular}{Base} = \frac{PM}{OM}$
4. The ratio of the hypotenuse to perpendicular, is called “cosecant of the angle θ” and it is written as $\text{cosec}\,\theta$.
   i.e. $\text{cosec}\,\theta = \frac{\text{hypotenuse}}{Perpendicular} = \frac{OP}{PM}$

5. The ratio of the hypotenuse to base, is called "secant of the angle θ" and it is written as $\sec\theta$.

   i.e. $\sec\theta = \frac{\text{Hypotenuse}}{Base} = \frac{OP}{OM}$.

6. The ratio of the base to perpendicular, is called "cotangent of the angle θ" and it is written as $\cot\theta$.

   i.e. $\cot\theta = \frac{Base}{Perpendicular} = \frac{OM}{PM}$.

Notes :

i**.** All the above six ratios are called trigonometrical ratios (T-Ratios).

ii. $cosec\theta = \frac{1}{\sin\theta}$. Or, $sin\,\theta$ and $cosec\theta$ are reciprocal ratios.

$sec\theta = \frac{1}{cos\theta}$, Or, $cos\,\theta$ and $sec\theta$ are reciprocal ratios.

$\cot\theta = \frac{1}{tan\theta}$, Or, $tan\theta$ and $\cot\theta$ are reciprocal ratios.

iii. For angle measure $\pi/2$, we define

$$sin\frac{\pi}{2} = 1, \cos\frac{\pi}{2} = 0, cot\frac{\pi}{2} = 0, cosec\frac{\pi}{2} = 1,$$

$tan\frac{\pi}{2}$ and $sec\frac{\pi}{2}$ are not defined.

$sin\frac{\pi}{2}, \cos\frac{\pi}{2}, cot\frac{\pi}{2}, cosec\frac{\pi}{2}$ are not defined as ratios of sides. So instead of using trigonometric ratios, we use a more general form for them, called as trigonometric functions, in due course. For the same reason we do not use the term trigonometric ratio for $sin0^{\circ}$, $\cos 0^{\circ}$, $tan0^{\circ}$and $sec0^{\circ}$. However we define

$$sin0^{\circ} = 0, \cos 0^{\circ} = 1, tan0^{\circ} = 0 \text{ and } sec0^{\circ} = 1.$$

Trigonometric functions:

The six trigonometric functions are given by the following.

(i) sine: $R \to [-1, 1]$,

(ii) $cosine: R \to [-1,1]$,

(iii) $tangent: R - \{(2n+1)\frac{\pi}{2} : n \in Z \to R$

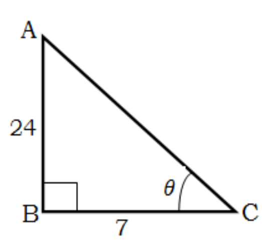
(iv) $cotangent: R - \{n\pi : n \epsilon Z\} \to R$

(v) $secant: R - \left\{(2n+1)\frac{\pi}{2} : n \epsilon Z\right\} \to R - (-1,1)$

(vi) $cosecant: R - \{n\pi : n\epsilon Z\} \to R - (-1, 1)$

For θ = (2n+1)$\frac{\pi}{2}$$: n \in Z$,tan$\theta$,$sec\ \theta$ does not exist. Similarly for $\theta = n\pi : n \in Z$ cot θ, cosec θ does not exist ,for this reason those terms are excluded from the respective functions.

Example-1: In ∆ABC, right angle is at B and AB=24 cm, BC=7cm.



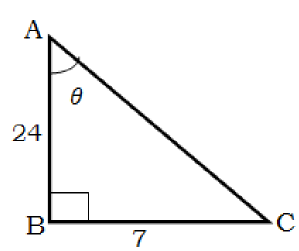


Fig. 1

Fig. 2.5 Fig 2.6

(i) In Fig-1, $\angle B = 90^{\circ}$ and $\angle C = \theta$ is the given angle.
So, Base $= b = BC = 7cm$, Hypotenuse $= h = AC$
and Perpendicular$= p = AB = 24cm$.
By using Pythagoras theorem, $p^2 + b^2 = h^2$
$\therefore h^2 = 24^2 + 7^2 = 625$ and $h = 25cm$
Therefore, $\sin A = \frac{p}{h} = \frac{24}{25}$, $\cos A = \frac{b}{h} = \frac{7}{25}$ and $\tan A = \frac{p}{b} = \frac{24}{7}$
(ii) But, In Fig-2, $\angle B = 90^{\circ}$ and $\angle A = \theta$ is the given angle.
So, Base $= b = AB = 7cm$, Hypotenuse $= h = AC$
and Perpendicular$= p = BC = 7cm$.
By using Pythagoras theorem, $p^2 + b^2 = h^2$
$\therefore h^2 = 24^2 + 7^2 = 625$ and $h = 25cm$
(iii) Therefore, $\sin A = \frac{p}{h} = \frac{7}{25}$, $\cos A = \frac{b}{h} = \frac{24}{25}$ and $\tan A = \frac{p}{b} = \frac{7}{24}$.

Example-2: Let $cot\theta = \frac{7}{8}$,
So, $cot\theta = \frac{b}{p} = \frac{7}{8} = k$, where k is a proportionality constant.
$\therefore b = 7k$ and $p = 8k$
By using Pythagoras theorem: $p^2 + b^2 = h^2$

$$\Rightarrow (8k)^2 + (7k)^2 = h^2$$
$$\Rightarrow h^2 = 113k^2 \;\; or \;\; h = \sqrt{113}k$$

Hence, $sec\theta = \frac{h}{b} = \frac{\sqrt{113}\,k}{7k} = \frac{\sqrt{113}}{7}$ and $cosec\,\theta = \frac{h}{p} = \frac{\sqrt{113}}{8}$.

Trigonometry Identity

(i) $\boldsymbol{sin^2\theta + cos^2\theta = 1}$

(ii) $\boldsymbol{sec^2\theta - tan^2\theta = 1}$

(iii) $\boldsymbol{cosec^2\theta - cot^2\theta = 1}$

Proof:

(i) LHS: $sin^2\theta + cos^2\theta = (sin\theta)^2 + (cos\theta)^2 = \left(\frac{p}{h}\right)^2 + \left(\frac{b}{h}\right)^2$
$= \frac{p^2}{h^2} + \frac{b^2}{h^2} = \frac{p^2+b^2}{h^2} = \frac{h^2}{h^2} = 1$ =RHS
[Note: By Pythagoras Theorem,
In a right angled triangle, $p^2 + b^2 = h^2$.]

(ii) LHS: $sec^2\theta - tan^2\theta = (sec\theta)^2 - (tan\theta)^2 = \left(\frac{h}{b}\right)^2 - \left(\frac{p}{b}\right)^2$
$= \frac{h^2}{b^2} - \frac{p^2}{b^2} = \frac{h^2-p^2}{b^2} = \frac{b^2}{b^2} = 1$=RHS

(iii) LHS: $cosec^2\theta - cot^2\theta = (cosec\theta)^2 - (cot\theta)^2 = \left(\frac{h}{p}\right)^2 - \left(\frac{b}{p}\right)^2$

$$= \frac{h^2}{p^2} - \frac{b^2}{p^2} = \frac{1}{p^2}(h^2 - b^2) = \frac{1}{p^2}(p^2) = 1\text{=RHS}$$

Note: All the above relations/identities hold good for any value of θ. i.e. these identities are independent of the angle(θ). i.e. whatever may be the angle, the relations are true.
Example : $sin^2x + cos^2x = 1$, $cosec^2\alpha - cot^2\alpha = 1$, $sec^2\beta - tan^2\beta = 1$, *etc.*

Some Solved Problems

Q.1: Prove $\frac{sin\theta}{1+cos\theta} + \frac{1+cos\theta}{sin\theta} = 2\ cosec\theta$

Proof:

LHS $= \frac{sin\theta}{1+cos\theta} + \frac{1+cos\theta}{sin\theta}$

$= \frac{sin\theta.sin\theta + (1+cos\theta)(1+cos\theta)}{(1+cos\theta)sin\theta}$

$= \frac{sin^2\theta + (1+cos\theta)^2}{(1+cos\theta)sin\theta}$

$= \frac{sin^2\theta + 1+cos^2\theta+2cos\theta}{(1+cos\theta)sin\theta}$ $[\therefore\ sin^2\theta + cos^2\theta = 1$

$= \frac{1+1+2cos\theta}{(1+cos\theta)sin\theta}$

$= \frac{2}{sin\theta} = 2cosec\theta = RHS$

Q-2: Prove $\frac{1}{1-sinx} + \frac{1}{1+sinx} = 2sec^2x$

Proof:

LHS$= \frac{1}{1-sinx} + \frac{1}{1+sinx}$

$= \frac{1(1+sinx)+1(1-sinx)}{(1-sinx)(1+sinx)}$

$= \frac{1+sinx+1-sinx}{1-sin^2x}$ $[\therefore\ (a-b)(a+b) = a^2 - b^2]$

$= \frac{2}{cos^2x}$ $[\therefore\ 1 - sin^2x = cos^2x$

$= 2cosec^2x$=RHS

Q-3: Prove $(cosec\ \theta - cot\theta)^2 = \frac{1-cos\theta}{1+cos\theta}$

Proof:

LHS$= (cosec\ \theta - cot\theta)^2$

$= \left(\frac{1}{sin\theta} - \frac{cos\theta}{sin\theta}\right)^2$

$= \left(\frac{1-cos\theta}{sin\theta}\right)^2$

$= \frac{(1-cos\theta)^2}{(1-cos\theta)(1+cos\theta)}$ $[\because sin^2\theta = 1-cos^2\theta = (1-cos\theta)(1+cos\theta)]$

$= \frac{1-cos\theta}{1+cos\theta}$ =RHS

Q-4: Prove $\frac{cosec\theta}{cosec\theta-1} + \frac{cosec\theta}{cosec\theta+1} = 2sec^2\theta$

Proof :

LHS = $\frac{cosec\theta}{cosec\theta-1} + \frac{cosec\theta}{cosec\theta+1}$

$= \frac{cosec\theta(cosec\theta+1)+cosec\theta(cosec\theta-1)}{(cosec\theta-1)(cosec\theta+1)}$

$= \frac{cosec^2\theta+cosec\theta+cosec^2\theta-cosec\theta}{cosec^2\theta-1}$

$= \frac{2cosec^2\theta}{cot^2\theta} = \frac{2}{sin^2\theta} \cdot \frac{sin^2\theta}{cos^2\theta} = 2sec^2\theta$=RHS

Q-5: Prove $\sqrt{\frac{1+sin\theta}{1-sin\theta}} = sec\theta + tan\theta$

Proof: LHS= $\sqrt{\frac{1+sin\theta}{1-sin\theta}} = \sqrt{\frac{1+sin\theta}{1-sin\theta}\,\frac{1+sin\theta}{1+sin\theta}} = \sqrt{\frac{(1+sin\theta)^2}{1-sin^2\theta}} = \sqrt{\frac{(1+sin\theta)^2}{cos^2\theta}}$

$= \sqrt{\left(\frac{1+sin\theta}{cos\theta}\right)^2} = \frac{1+sin\theta}{cos\theta} = \frac{1}{cos\theta} + \frac{sin\theta}{cos\theta} = sec\theta + tan\theta$ =RHS

Q-6: Prove $\frac{tanx}{1-cotx} + \frac{cotx}{1-tanx} = secx.cosecx + 1$

Proof:

LHS= $\frac{tanx}{1-cotx} + \frac{cotx}{1-tanx}$

$= \frac{sinx}{cosx}\frac{1}{1-\frac{cosx}{sinx}} + \frac{cosx}{sinx}\frac{1}{1-\frac{sinx}{cosx}}$

$= \frac{sinx}{cosx}\frac{1}{\frac{sinx-cosx}{sinx}} + \frac{cosx}{sinx}\frac{1}{\frac{cosx-sinx}{cosx}}$

$= \frac{sin^2x}{cosx(sinx-cosx)} - \frac{cos^2x}{sinx(sinx-cosx)}$

$= \frac{sin^3x-cos^3x}{sinx\,cosx\,(sinx-cosx)}$ $\left[\begin{array}{c} \because sin^3x - cos^3x, a^3 - b^3 = (a-b)(a^2+ab+b^2) \\ = (sinx - cosx)(sin^2x + cos^2x + sinx\,cosx) \\ = (sinx - cosx)(1 + sinx.cosx) \end{array}\right]$

$= \frac{(sinx-cosx)(1+sinx.cosx)}{sinx\,cosx\,(sinx-cosx)}$

$= \frac{1}{sinx\,cosx} + 1 = secx\,cosecx + 1$ =RHS

Q-7: Prove $\frac{sec\theta-tan\theta}{sec\theta+tan\theta} = 1 - 2sec\theta.tan\theta + 2tan^2\theta$

Proof:

LHS: $\frac{sec\theta - tan\theta}{sec\theta + tan\theta}$

$= \frac{sec\theta - tan\theta}{sec\theta + tan\theta} \cdot \frac{sec\theta - tan\theta}{sec\theta - tan\theta}$

$= \frac{(sec\theta - tan\theta)^2}{sec^2\theta - tan^2\theta}$ $[\because sec^2\theta - tan^2\theta = 1]$

$= \frac{(sec\theta - tan\theta)^2}{1}$

$= (1 + tan^2\theta) + tan^2\theta - 2sec\theta.tan\theta$

$= 1 - 2sec\theta.tan\theta\ + 2tan^2\theta$=RHS

Q-8: Prove that $sin^8\theta - cos^8\theta = (sin^2\theta - cos^2\theta)(1 - 2sin^2\theta.cos^2\theta)$

Proof:

LHS=$sin^8\theta - cos^8\theta$

$= (sin^4\theta)^2 - (cos^4\theta)^2$

$= (sin^4\theta - cos^4\theta)(sin^4\theta + cos^4\theta)$ $[As\ a^2 - b^2 = (a - b)(a + b)]$

$= \{(sin^2\theta)^2 - (cos^2\theta)^2\}\{(sin^2\theta)^2 + (cos^2\theta)^2\}$

$= (sin^2\theta - cos^2\theta)(sin^2\theta + cos^2\theta)\{(sin^2\theta + cos^2\theta)^2 - 2sin^2\theta cos^2\theta\}$

$[\because a^2 + b^2 = (a + b)^2 - 2ab$ and $sin^2\theta + cos^2\theta = 1]$

$= (sin^2\theta - cos^2\theta)(1 - 2sin^2\theta cos^2\theta)$ =RHS

Signs of T-Ratios

Let a revolving line OP, starting from OX trace and $\angle XOP = \theta$. From P, Draw $PM \perp XOX'$.
In the right angled triangle, $OP^2 = OM^2 + PM^2$,
$OP = +\sqrt{OM^2 + PM^2}$

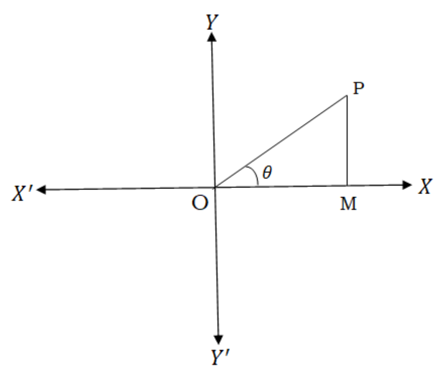


Fig 2.7

Suppose the point 'P' lies in 1st quadrant.
OM=+ve, PM=+ve and OP=+ve

$sin\theta = \frac{PM}{OP} = \frac{+ve}{+ve} = +$ve

$cos\theta = \frac{OM}{OP} = \frac{+ve}{+ve} = +$ve

$tan\theta = \frac{PM}{OM} = \frac{+ve}{+ve} = +$ve

In 1st quadrant, all the T-ratios are having positive signs.

Suppose the point 'P' lies in 2nd quadrant.
OM= $-ve$, PM= $+ve$ and OP= $+ve$

$sin\theta = \frac{PM}{OP} = \frac{+ve}{+ve} = +$ve

$cos\theta = \frac{OM}{OP} = \frac{-ve}{+ve} = -\text{ve}$

$tan\theta = \frac{PM}{OM} = \frac{+ve}{-ve} = -\text{ve}$

In 2nd quadrant, only $sin\theta$ and $cosec\theta$ are positive and all other T-ratios are having negative signs.

Suppose the point 'P' lies in 3rd quadrant.

OM= $-ve$, PM= $-ve$ and $OP = +ve$

$sin\theta = \frac{PM}{OP} = \frac{-ve}{+ve} = -\text{ve}$

$cos\theta = \frac{OM}{OP} = \frac{-ve}{+ve} = -\text{ve}$

$tan\theta = \frac{PM}{OM} = \frac{-ve}{-ve} = +\text{ve}$

In 3rd quadrant, only $tan\theta$ and $cot\theta$ are positive and all other T-ratios are having negative signs.

Suppose the point 'P' lies in 4th quadrant.

OM= $+ve$, PM= $-ve$ and $OP = +ve$

$sin\theta = \frac{PM}{OP} = \frac{-ve}{+ve} = -\text{ve}$

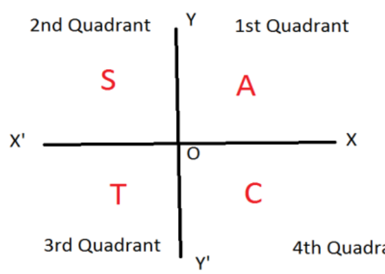
$cos\theta = \frac{OM}{OP} = \frac{+ve}{+ve} = +\text{ve}$

$tan\theta = \frac{PM}{OM} = \frac{-ve}{+ve} = -\text{ve}$

In 4th quadrant, only $cos\theta$ and $sec\theta$ are positive and all other T-ratios are having negative signs.

Notes: ASTC-Rule

i. In 1st quadrant, all T-ratios are $+ve$.
ii. In 2nd quadrant, sine is +ve and all others –ve
iii. In 3rd quadrant, tangent is +ve and all others –ve.
iv. In 4th quadrant, cosine is +ve and all others –ve.
v. The sign of any T-Ratio in any quadrant can be recalled by the words 'all-sin-tan-cos' or 'add sugar to coffee' and this rule is known as 'ASTC-Rule'. Whatever is written in a particular quadrant, this T-ratio along with its reciprocal are positive and all other ratios are negative.



ASTC - Rule

Trigonometric Ratio of Selected Angles

The values of T-ratios for some selected angles like $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$ are given below..

Angles $\rightarrow$	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
$\text{cosec}\,\theta$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Some Solved Problems

Q-1: Find the value of $\frac{5cos^2 60° + 4sec^2 30^{\circ} - tan^2 45°}{sin^2 30° + cos^2 30°}$

Sol:

Using the trigonometric values,

$$\frac{5cos^2 60° + 4sec^2 30^{\circ} - tan^2 45°}{sin^2 30° + cos^2 30°} = \frac{5\left(\frac{1}{2}\right)^2 + 4\left(\frac{2}{\sqrt{3}}\right)^2 - 1^2}{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{\frac{5}{4} + \frac{16}{3} - 1}{\frac{1}{4} + \frac{3}{4}} = \frac{\frac{15+64-12}{12}}{\frac{4}{4}} = \frac{67}{12}$$

Q-2: Find the value of $\frac{\cot 45°}{\sqrt{1-cot^2 60°}}$

Sol:

As we know that, $\cot 45^{\circ} = 1$ and $\cot 60^{\circ} = \frac{1}{\sqrt{3}}$

Therefore, $\frac{\cot 45°}{\sqrt{1-cot^2 60°}} = \frac{1}{\sqrt{1-(1/\sqrt{3})^2}} = \frac{1}{\sqrt{2/3}} = \frac{\sqrt{3}}{\sqrt{2}}$

Limits of the values of T-ratios:

1. $-1 \leq \sin\theta \leq 1$, and $-1 \leq \cos\theta \leq 1$
   i.e. the minimum values of both sine and cosine of angles are -1 and maximum values of both sine and cosine of angles are $+1$.
2. $\text{cosec}\,\theta$ and $\sec\theta$ each cannot be numerically less than unity,
   i.e. $-1 \geq cosec\ \theta$ or $cosec\ \theta \geq 1$ and $-1 \geq sec\ \theta$ or $sec\ \theta \geq 1$
3. $\tan\theta$ and $\cot\theta$ can have any numerical value,
   $\tan\theta \in R$ and $\cot\theta \in R$.

Some Solved Problems

Q-1: Find the maximum and minimum value of $5\ sinx + 12\ cosx$.

Sol:

Let $5 = \text{r} \cos\alpha$ and $12 = \text{r} \sin\alpha$.

By squaring and adding,

$5^2 + 12^2 = (\text{r} \cos\alpha)^2 + (\text{r} \sin\alpha)^2$

$\Rightarrow 25 + 144 = r^2(cos^2\alpha + sin^2\alpha)$

$\Rightarrow 169 = r^2.1 = r^2$

$\Rightarrow r = 13$

Now, the given expression $5\ \text{sinx} + 12\ \text{cosx}$ can be reduced to

$\text{r cos}\alpha\ \ \text{sin}x + \text{r sin}\alpha\ \text{cos}x$

$= r(sinx\ cos\alpha + cosx\ sin\alpha)$

$= r\sin(x+\alpha) = 13\, sin(x+\alpha)$

We know that the minimum and maximum values of $\sin\theta$ are -1 and 1 respectively, i.e. $-1 \le \sin\theta \le 1$

Therefore $\quad -1 \le sin(x+\alpha) \le 1$

Or, $-13 \le 13\, sin(x+\alpha) \le 13$

Hence, the maximum and minimum values of $5\text{sinx} + 12\ \text{cosx}$ are 13 and −13.

Q-2: Find the maximum value of $2 + 3sinx + 4cosx$.

Sol:

Let $3 = rcos\alpha$ and $4 = rsin\alpha$

So, $r = \sqrt{3^2 + 4^2} = 5$

Now, $3sinx + 4cosx = r\cos\alpha\sin x + \ r\sin\alpha\cos x$

$= r(sinx\ cos\alpha + cosx\ sin\alpha) = 5\sin(x+\alpha)$

We know that, the maximum value of $sin\theta$ =1

∴ Maximum value of $\sin(x+\alpha) = 1$

⇒ Maximum value of $5\sin(x+\alpha) = 5.1 = 5$

⇒ Maximum value of $2 + 5\sin(x+\alpha) = 2 + 5 = 7$

Values of T-ratios of allied angles

1. **T-ratios of $(-\theta)$ in terms of θ, for all values of θ.**

   Let OX be the initial line. Let OP be the position of the radius vector after tracing an angle θ in the anticlockwise sense which we take as positive sense. Let OP' be the position of the radius vector after tracing (θ) in the clockwise sense, which we take as negative sense. So $\angle P'OX$ will be taken as $(-\theta)$. Join PP'. Let it meet OX at M.

   Here, $\Delta OPM = \Delta P'OM,\ \angle P'OM = -\theta,\ OP' = OP,\ P'M = -PM$

   Now $sin(-\theta) = \frac{P'M}{OP'} = \frac{-PM}{OP} = -sin\theta$

   $cos(-\theta) = \frac{OM}{OP'} = \frac{OM}{OP} = cos\theta$

   $tan(-\theta) = \frac{P'M}{OM} = \frac{-PM}{OM} = -tan\theta$

   Similarly,

   $cot(-\theta) = \frac{OM}{P'M} = \frac{OM}{-PM} = -cot\theta$

   $sec(-\theta) = \frac{OP'}{OM} = sec\theta$

   $cosec(-\theta) = \frac{OP'}{P'M} = \frac{OP}{-PM} = -cosec\ \theta$

   Example:

   $\sin(-60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$

   $\cos(-30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$

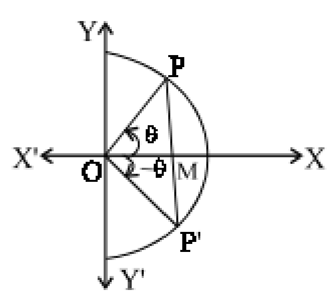


Fig 2.8

$\tan(-45^{\circ}) = -\tan 45^{\circ} = -1$

2. T-ratios of $(90-\theta)$ in terms of θ, for all the values of θ

Let OPM be a right angled triangle with ∠ POM = 90°,

∠OMP =θ, ∠OPM =90°- θ

$sin(90° - \theta) = \frac{OM}{PM} = cos\ \theta$

$cos(90° - \theta) = \frac{OP}{PM} = sin\ \theta$

$tan\ (90° - \theta) = \frac{OM}{OP} = cot\theta$

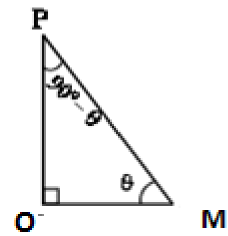


Fig 2.9

Similarly,

$\cot(90-\theta) = +\tan\theta$

$\sec(90-\theta) = +\operatorname{cosec}\theta$

$\operatorname{cosec}(90-\theta) = +\sec\theta$

Here, the angle θ and $90-\theta$ are called complementary angles and each of the angle is called complement of each other.

Example:

$\sin(90^{\circ}-30^{\circ}) = +\cos 30^{\circ} = \frac{\sqrt{3}}{2}$

$\cos(90^{\circ}-60^{\circ}) = +\sin 60^{\circ} = \frac{\sqrt{3}}{2}$

$\tan(90^{\circ}-45^{\circ}) = +\cot 45^{\circ} = 1$

3. T-ratios of $(90^{\circ}+\theta)$ in terms of θ, for all the values of θ

Let $\angle POX = \theta$ and $\angle P'OX = 90° + \theta$. Draw PM and P'M' perpendiculars to the X-axis.

Now ΔPOM ≅Δ P'OM'

∴ P'M' = OM and OM' = –PM

Now $\sin(90°+\theta) = \frac{PM}{OP} = \frac{P'M'}{OP'} = cos\theta$

$\cos(90°+\theta) = \frac{OM'}{OP'} = \frac{-PM}{0P} = -sin\theta$

$\tan(90°+\theta) = \frac{P'M'}{OM'} = \frac{-OM}{PM} = -cot\theta$

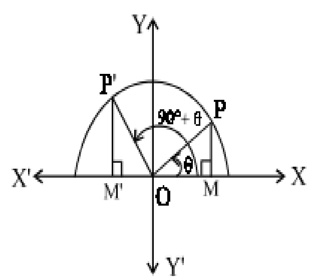


Fig 2.10

Similarly,

$\cot(90°+\theta) = -\tan\theta$

$\sec(90°+\theta) = -\operatorname{cosec}\theta$

$\operatorname{cosec}(90°+\theta) = +\sec\theta$

Examples:

$\sin 120° = \sin(90°+30°) = +\cos 30° = \frac{\sqrt{3}}{2}$

$\cos 150° = \cos(90°+60°) = -\sin 60° = -\frac{\sqrt{3}}{2}$

$\tan 135° = \tan(90°+45°) = -\cot 45° = -1$

$\sec 150° = \sec(90°+60°) = -cosec60° = -\frac{2}{\sqrt{3}}$

Similarly, the values of T-ratios for following allied angles can also be proved.

4. T-ratios of $(180° - \theta)$ in terms of θ, for all the values of θ

$\sin(180° - \theta) = +\sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$
$\cot(180° - \theta) = -\cot\theta$
$\sec(180° - \theta) = -\sec\theta$
$\operatorname{cosec}(180° - \theta) = +\operatorname{cosec}\theta$

Here, the angle θ and $180° - \theta$ are called supplementary angles and each of the angle is called supplement of each other.

Examples:

$\cos 150° = \cos(180° - 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$
$\sec 120° = \sec(180° - 60°) = -\sec 60° = -2$
$\cot 135° = \cot(180° - 45°) = -\cot 45° = -1$

5. T-ratios of $(180° + \theta)$ in terms of θ, for all the values of θ

$\sin(180° + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = +\tan\theta$
$\cot(180° + \theta) = +\cot\theta$
$\sec(180° + \theta) = -\sec\theta$
$\operatorname{cosec}(180° + \theta) = -\operatorname{cosec}\theta$

Examples:

$\sin 210° = \sin(180° + 30°) = -\sin 30° = -\frac{1}{2}$
$\cos 225° = \cos(180° + 45°) = -\cos 45° = -\frac{1}{\sqrt{2}}$
$\tan 240° = \tan(180° + 60°) = +\tan 60° = \sqrt{3}$

6. T-ratios of $(270° - \theta)$ in terms of θ, for all the values of θ

$\sin(270° - \theta) = -\cos\theta$
$\cos(270° - \theta) = -\sin\theta$
$\tan(270° - \theta) = +\cot\theta$
$\cot(270° - \theta) = +\tan\theta$
$\sec(270° - \theta) = -\operatorname{cosec}\theta$
$\operatorname{cosec}(270° - \theta) = -\sec\theta$

Examples:

$\sin 210° = \sin(270° - 60°) = -\cos 60° = -\frac{1}{2}$
$\cos 240° = \cos(270° - 30°) = -\sin 30° = -\frac{1}{2}$
$\tan 225° = \tan(270° - 45°) = +\cot 45° = +1$

7. T-ratios of $(270° + \theta)$ in terms of θ, for all the values of θ

$\sin(270° + \theta) = -\cos\theta$
$\cos(270° + \theta) = +\sin\theta$

$\tan(270° + \theta) = -\cot\theta$
$\cot(270° + \theta) = -\tan\theta$
$\sec(270° + \theta) = +\text{cosec}\,\theta$
$\text{cosec}(270° + \theta) = -\sec\theta$
Examples:
$\sin 315^{\circ} = \sin(270° + 45°) = -\cos 45° = -\frac{1}{\sqrt{2}}$
$\cos 300^{\circ} = \cos(270° + 30°) = \sin 30^{\circ} = \frac{1}{2}$
$\cot 330^{\circ} = \cot(270° + 60°) = -\tan 60^{\circ} = -\sqrt{3}$

8. **T-ratios of $(\mathbf{360°} - \boldsymbol{\theta})$ in terms of $\boldsymbol{\theta}$, for all the values of $\boldsymbol{\theta}$**
$\sin(360° - \theta) = -\sin\theta$
$\cos(360° - \theta) = +\cos\theta$
$\tan(360° - \theta) = -\tan\theta$
$\cot(360° - \theta) = -\cot\theta$
$\sec(360° - \theta) = +\sec\theta$
$\text{cosec}(360° - \theta) = -\text{cosec}\,\theta$
Note: T-ratios of $(360° - \theta)$ and those of $(-\theta)$ are the same.
Examples:
$\sin 330^{\circ} = \sin(360° - 30°) = -\sin 30° = -\frac{1}{2}$
$\sec 300^{\circ} = \sec(360° - 60°) = \sec 60° = 2$
$co\sec 315^{\circ} = \text{cosec}(360° - 45°) = -cosec 45° = -\sqrt{2}$

9. **T-ratios of $(\mathbf{360°} + \boldsymbol{\theta})$ in terms of $\boldsymbol{\theta}$, for all the values of $\boldsymbol{\theta}$**
$\sin(360° + \theta) = +\sin\theta$
$\cos(360° + \theta) = +\cos\theta$
$\tan(360° + \theta) = +\tan\theta$
$\cot(360° + \theta) = +\cot\theta$
$\sec(360° + \theta) = +\sec\theta$
$\text{cosec}(360° + \theta) = +\text{cosec}\,\theta$

Note:

1. T-ratios of $(360° + \theta)$ or $(2\pi + \theta)$ and those of θ are same.
2. T-ratios of $(n \times 360° + \theta)$, where $n = 1, 2, 3, \ldots$ also will be the same as that of θ.
3. In general, $\sin(n\pi + \theta) = (-1)^n sin\theta$
   $\cos(n\pi + \theta) = (-1)^n cos\theta$
   $\tan(n\pi + \theta) = tan\theta$
4. Similarly, $\sin\left(n\frac{\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}} cos\theta$
   $\cos\left(n\frac{\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}} sin\theta$
   $\tan\left(n\frac{\pi}{2} + \theta\right) = -cot\theta$, where n is any odd integer

Some Solved Problems

Q-1 : State $cos302^{\circ}$ is positive or negative.

Sol:

$\cos(3\times 90^{\circ}+32^{\circ}) = sin32°$

$\therefore \sin 32^{\circ}$ lies in 1st quadrant, and by ASTC rule, all T-ratios are positive in 1st quadrant.

So, $\cos 302°$ is positive sign.

Q-2 : Find the value of $sin1230^{\circ}$.

Sol:

$sin1230^{\circ} = \left(\sin 3\times 360^{\circ} + 150^{\circ}\right) = sin150^{\circ}$

$= \sin(180^{\circ} - 30^{\circ}) = \sin\ 30° = 1/2$

Q-3: Express $sin\ 1185^{\circ}$ as the trigonometric ratio of some acute angle**.**

Sol:

$\sin 1185° = sin(13\ \times\ 90°\ \ + 15°)$

$= (-1)^{\frac{13-1}{2}} \cos 15° = (-1)^6\ cos15° = cos15°$

Q-4 : Find the value of $\log tan17° + logtan37° + logtan53° + logtan73°$

Sol:

$\log tan17° + logtan37° + logtan53° + logtan73°$

$= \log tan17°\ tan37°\ tan53°\ tan73°$

$= logtan17°\ tan37°\ \tan(90° - 37°)\ \tan(90° - 17°)$

$= logtan17°tan37°cot37°cot17°$

$= log1 = 0$ [$\because\ tan17°cot17° = 1\ and\ tan37°cot37° = 1$]

Q-5: Show that $\frac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°-\theta)} = -1.$

Proof:

LHS= $\frac{cos(90°+\theta)\,sec(-\theta)\,tan(180°-\theta)}{sec(360°-\theta)\,sin(180°+\theta)\,cot(90°-\theta)} = \frac{-sin\theta\ .\ sec\theta\ \ (-tan\theta)}{sec\theta\ \ (-sin\theta)\ \ tan\theta} = -1$

Even Function

A function $f(x)$ is said to be an even function of x, if $f(-x) = f(x)$.

Examples:

1. Let $f(x) = \cos x$, then $f(-x) = \cos(-x) = \cos x$

$\therefore\ f(x) = f(-x) = \cos x$.

Hence, $f(x) = \cos x$ is an even function.

2. Let $f(x) = \sin x \tan x$,

then $f(-x) = sin(-x)\,tan(-x) = (-\sin x)(-\tan x) = \sin x \tan x$

$\therefore\ f(x) = f(-x) = sin\,x\,tan\,x$.

Hence, $f(x) = sin\,x\,tan\,x$ is an even function.

3. Let $f(x) = 1 + x^4 + \cot^2 x$,

then $f(-x) = 1 + (-x)^4 + \{cot(-x)\}^2 = 1 + x^4 + (-\,cot\,x)^2$

$= 1 + x^4 + cot^2\,x$

$\therefore\ f(x) = f(-x) = 1 + x^4 + cot^2\, x.$

Hence, $f(x) = 1 + x^4 + cot^2\, x$ is an even function.

4. Let $f(x) = cos2x$

Then, $f(-x) = \cos 2(-x) = \cos 2x = f(x)$ [As $\cos(-\theta) = \cos\theta$]

So, $f(x) = cos2x$ is an even function

Odd Function

A function $f(x)$ is said to be an odd function of x, if $f(-x) = -f(x)$.

Example:

1. Let $f(x) = \sin x$, then $f(-x) = \sin(-x) = -\sin x = -f(x)$

So, $f(x) = \sin x$ is an odd function.

2. Let $f(x) = \tan x$, then $f(-x) = \tan(-x) = -\tan x = -f(x)$

So, $f(x) = \tan x$ is an odd function.

3. Let $f(x) = x^3 + cosec\ x$,

Then, $f(-x) = (-x)^3 + cosec\left(-x\right) = (-x^3) + (-cosec\ x)$

$$= -(x^3 + cosec\ x) = -f(x)$$

Hence, $f(x) = x^3 + cosec\ x$ is an odd function.

4. But, $Let\ f(x) = sin3x + 5$

Then, $f(-x) = sin(-3x) + 5 = -sin\,3x + 5$

Here, $f(-x)$ expressed neither as $f(x)$ nor as $-f(x)$.

Hence, $f(x) = sin3x + 5$ is neither an odd function nor an even function.

Theorem-1: (Addition Theorems)

(i) $\boldsymbol{sin(A+B) = sinA\ cosB + cosA\ sinB}$

(ii)

(iii) $\boldsymbol{tan(A+B) = \frac{tanA + tanB}{1 - tanA\ tanB}}$

(iv) $\boldsymbol{\cot(A+B) = \frac{cotA\ cotB - 1}{cotB + cotA}}$

Proof : Let the revolving line OM starting from the line OX make an angle $\angle$XOM=A and then further move to make $\angle MON = B,$

So that $\angle XON = A + B$

Let "P" be any point on the line ON.

Draw $PR \perp OX$, $PT \perp OM$, $TQ \perp PR$ and $TS \perp OX$

Then $\angle QPT = 90° - \angle PTQ = \angle QTO = \angle XOM = A$

We have from Δ OPR

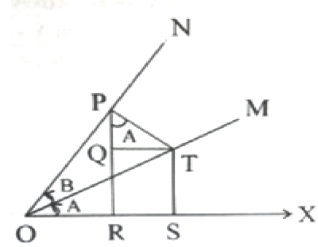


Fig 2.11

(i) $sin(A+B) = \frac{RP}{OP} = \frac{QR+PQ}{OP} = \frac{TS+PQ}{OP}$ $(\because QR = TS)$

$$= \frac{TS}{OP} + \frac{PQ}{OP} = \left(\frac{TS}{OT}\right)\left(\frac{OT}{OP}\right) + \left(\frac{PQ}{PT}\right)\left(\frac{PT}{OP}\right)$$

$$= sinA\ cosB + cosA\ sinB$$

(ii) $\cos(A+B) = \frac{QR}{OP} = \frac{OS-RS}{OP} = \frac{OS}{OP} - \frac{RS}{OP}$

$$= \left(\frac{OS}{OT}\right)\left(\frac{OT}{OP}\right) - \left(\frac{QT}{PT}\right)\left(\frac{PT}{OP}\right)$$

$$= cosA\ cosB - sinA\ sinB$$

(iii) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{sinA\,cosB + cosA\,sinB}{cosA\,cosB - sinA\,sinB}$

Dividing numerator and denominator by $\cos A \,.\, \cos B$

$$= \frac{\frac{sinAcosB}{cosAcosB} + \frac{cosAsinB}{cosAcosB}}{\frac{cosAcosB}{cosAcosB} - \frac{sinAsinB}{cosAcosB}} = \frac{tanA + tanB}{1 - tanA\,tanB}$$

Similarly, we can prove the followings theorems.

Theorem-2

(i) $\boldsymbol{\sin(A-B) = sinA\,cosB - cosA\,sinB}$

(ii) $\boldsymbol{\cos(A-B) = \cos A\,cosB + sinA\,sinB}$

(iii) $\boldsymbol{\tan(A-B) = \frac{tanA - tanB}{1 + tanA\,tanB}}$

The above theorems can be proved, by replacing B with –B in theorem-1.

Theorem-3

$\boldsymbol{\tan(A+B+C) = \frac{tanA + tanB + tanC - tanA\,tanB\,tanC}{1 - tanA\,tanB - tanB\,tanC - tanC\,tanA}}$ **, for** $\boldsymbol{A, B, C \in R}$

Theorem-4

(i) $\boldsymbol{sin(A+B)\,sin(A-B) = sin^2A - sin^2B = cos^2B - cos^2A}$

(ii) $\boldsymbol{cos(A+B)\,cos(A-B) = cos^2A - sin^2B = cos^2B - sin^2A}$

Proof :

(i) $sin(A+B)\,sin(A-B)$

$= (\sin A \cos B + \cos A \sin B)(sin\,A\,cos\,B - cos\,A\,sin\,B)$

$= sin^2\,A\,cos^2\,B - cos^2\,A\,sin^2\,B$

$= sin^2\,A\,(1 - sin^2\,B) - (1 - sin^2\,A)\,sin^2\,B$

$= sin^2\,A - sin^2\,A\,sin^2\,B - sin^2\,B + sin^2\,A\,sin^2\,B$

$= sin^2\,A - sin^2\,B$ (Proved)

$= (1 - cos^2\,A) - (1 - cos^2\,B)$

$= cos^2\,B - cos^2\,A$ (Proved)

(ii) $cos(A+B)\,cos(A-B)$

$= (\cos A \cos B - \sin A \sin B)(cos\,A\,cos\,B + sin\,A\,sin\,B)$

$= cos^2\,A\,cos^2\,B - sin^2\,A\,sin^2\,B$

$= cos^2\,A\,(1 - sin^2\,B) - (1 - cos^2\,A)\,sin^2\,B$

$= cos^2\,A - cos^2\,A\,sin^2\,B - sin^2\,B + cos^2\,A\,sin^2\,B$

$= cos^2\,A - sin^2\,B$ (Proved)

$= (1 - sin^2\,A) - (1 - cos^2\,B)$

$= cos^2B - sin^2A$ (Proved)

Some Solved Problems

Q.1: Find the value of $\cos 15°$ **.**

Sol:

$cos15° = \cos(45° - 30°) = cos45°\ \ cos30° + sin45°\ \ sin30°$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Q-2: Find the value of $cos50°\ cos40° - sin50°\ sin40°$.

Sol:

$cos50°\ cos40° - sin50°\ sin40° = \cos(50° + 40°) = cos90° = 0$ [Use $\cos(A + B)$]

Q-3: If $\tan A = \frac{1}{2}$ and $tanB = \frac{1}{3}$, find the value of $\tan(A + B)$**.**

Sol:

We know that, $\tan(A + B) = \frac{tanA+tanB}{1-tanAtanB} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \frac{5/6}{5/6} = 1$.

Q-4: Prove that $\frac{cos9^\circ + sin9^\circ}{cos9^\circ - sin9^\circ} = tan54^\circ$

Proof:

LHS $= \frac{cos9^\circ+sin9^\circ}{cos9^\circ-sin9^\circ} = \frac{\frac{cos9^\circ}{cos9^\circ}+\frac{sin9^\circ}{cos9^\circ}}{\frac{cos9^\circ}{cos9^\circ}-\frac{sin9^\circ}{cos9^\circ}}$ (Dividing throughout by cos9°)

$$= \frac{1+tan9^\circ}{1-tan9^\circ} = \frac{tan45^\circ+tan9^\circ}{1-tan45^\circ tan9^\circ} \quad [\because tan45^\circ = 1]$$

$= ta\,n(45^\circ + 9^\circ) = tan54^\circ$ =RHS

Q-5: Prove that $sinA\ \sin(B - C) + sinB\ sin(C - A) + sinC\ sin(A - B) = 0$

Proof:

L.H.S: $sinA \sin(B - C) + sinB\ sin(C - A) + sinC\ sin(A - B)$

$= sinA(sinB\ cosC - cosB\ sinC) + sinB(sinC\ cosA - cosC\ sinA)$

$+sinC(sinA\ cosB - cosAsinB)$

$= sinA\ sinB\ cosC - sinA\ cosB\ sinC + sinB\ sinC\ cosA - sinBcosC\ sinA$

$+sinCsin - sinC\ cosA\ sinB$

(All the terms are cancelled with each other)

$= 0$ =R.H.S

Transformation of a Product into a Sum or Difference, and Vice-versa

(i) $\mathbf{\sin(A + B) + \sin(A - B) = 2 \sin A \cos B}$

(ii) $\mathbf{\sin(A + B) - \sin(A - B) = 2 \cos A \sin B}$

(iii) $\mathbf{\cos(A + B) + \cos(A - B) = 2 \cos A \cos B}$

(iv) $\mathbf{\cos(A + B) - \cos(A - B) = -2 \sin A \sin B}$

Proof:

From above established theorems 1 and 2,

(i) $sin(A+B) = sinA\ cosB + cosA\ sinB$
(ii) $\sin(A-B) = sinA\ cosB - cosA\ sinB$
(iii) $\cos(A+B) = \text{cosA cosB} - \text{sinA sinB}$
(iv) $\cos(A-B) = \cos A\ cosB + sinA\ sinB$

Adding (i) and (ii),

$\sin(A+B) + \sin(A-B) = (sinA\ cosB + cosA\ sinB) + (sinA\ cosB - cosA\ sinB)$
$= 2\sin A\cos B$

Subtracting (ii) from (i),

$\sin(A+B) - \sin(A-B) = (sinA\ cosB + cosA\ sinB) - (sinA\ cosB - cosA\ sinB)$
$= 2\cos A\sin B$

Again, Adding (iii) and (iv),

$\cos(A+B) + \cos(A-B) = (\text{cosA cosB} - \text{sinA sinB}) + (\cos A\ cosB + sinA\ sinB)$
$= 2\cos A\cos B$

Subtracting (iv) from (iii),

$\cos(A+B) - \cos(A-B) = (\text{cosA cosB} - \text{sinA sinB}) - (\cos A\ cosB + sinA\ sinB)$
$= -2\sin A\sin B$ (Proved)

Note :

Let $A+B=C$ and $A-B=D$

Then, $2A = C+D$ or $A = \frac{C+D}{2}$

and $2B = C-D$ or $B = \frac{C-D}{2}$

Putting the above values of A and B, in above four formulae, we get

$$\mathbf{\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)}$$
$$\mathbf{\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)}$$
$$\mathbf{\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)}$$
$$\mathbf{\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)}$$

Some Solved Problems

Q-1 : Prove that $sin50° - sin70° + sin10° = 0$.

Proof:

L.H.S= $sin50° - sin70° + sin10°$
$= \sin(60° - 10°) - \sin(60° + 10°) + sin10°$
$= -2cos60°\ sin10° + sin10°$ $(\because \sin(A-B) - \sin(A+B) = 2cosA\ sinB)$
$= -2 \times \frac{1}{2}\ sin10° + sin10°$
$= -sin10° + sin10° = 0$ =RHS

Q-2: Prove $\sin 10^{\circ} + \sin 20^{\circ} + \sin 40^{\circ} + \sin 50^{\circ} = \sin 70^{\circ} + \sin 80^{\circ}$

Proof:

LHS $= \sin 10^{\circ} + \sin 20^{\circ} + \sin 40^{\circ} + \sin 50^{\circ} = \sin 70^{\circ} + \sin 80^{\circ}$
$= (\sin 10^{\circ} + \sin 50^{\circ}) + (\sin 20^{\circ} + \sin 40^{\circ})$

$= 2\sin\left(\frac{10^\circ + 50^\circ}{2}\right)\cos\left(\frac{10^\circ - 50^\circ}{2}\right) + 2\sin\left(\frac{20^\circ + 40^\circ}{2}\right)\cos\left(\frac{20^\circ - 40^\circ}{2}\right)$
$= 2\sin 30^\circ \cos 20^\circ + 2\sin 30^\circ \cos 10^\circ$
$= 2\sin 30^\circ\left(\cos 20^\circ + \cos 10^\circ\right)$
$= 2\ \frac{1}{2}\left(\cos 20^\circ + \cos 10^\circ\right)$
$= \cos 20^\circ + \cos 10^\circ$
$= \cos\left(90^\circ - 70^\circ\right) + \cos\left(90^\circ - 80^\circ\right)$
$= \cos 70^\circ + \cos 80^\circ$ =RHS

Q-3: Prove that $\frac{cos7\alpha+cos3\alpha-cos5\alpha-cos\alpha}{sin7\alpha-sin3\alpha-sin5\alpha+sin\alpha} = cot2\alpha$
Proof:
LHS.: $= \frac{cos7\alpha+cos3\alpha-cos5\alpha-cos\alpha}{sin7\alpha-sin3\alpha-sin5\alpha+sin\alpha}$
$= \frac{(cos7\alpha+cos3\alpha)-(cos5\alpha+cos\alpha)}{(sin7\alpha-sin3\alpha)-(sin5\alpha-sin\alpha)}$
$= \frac{2\cos\left(\frac{7\alpha+3\alpha}{2}\right)\cos\left(\frac{7\alpha-3\alpha}{2}\right)-2\cos\left(\frac{5\alpha+\alpha}{2}\right)\cos\left(\frac{5\alpha-\alpha}{2}\right)}{2\cos\left(\frac{7\alpha+3\alpha}{2}\right)\sin\left(\frac{7\alpha-3\alpha}{2}\right)-2\cos\left(\frac{5\alpha+\alpha}{2}\right)\sin\left(\frac{5\alpha-\alpha}{2}\right)}$
$= \frac{2cos4\alpha\ cos2\alpha-2cos3\alpha\ cos2\alpha}{2cos4\alpha\ sin2\alpha-2cos3\alpha\ sin2\alpha}$
$= \frac{2cos2\alpha(cos4\alpha-cos3\alpha)}{2sin2\alpha(cos4\alpha-cos3\alpha)}$
$= \frac{cos2\alpha}{sin2\alpha} = cot2\alpha =$ RHS

Q-4: If $sinA = KsinB$, Prove that $tan\frac{1}{2}(A-B) = \frac{K-1}{K+1}\ tan\frac{1}{2}(A+B)$.
Proof:
Given $sinA = KsinB$
$\Rightarrow \frac{sinA}{sinB} = \frac{K}{1}$
$\Rightarrow \frac{sinA - sinB}{sinA + sinB} = \frac{K-1}{K+1}$
$\Rightarrow \frac{2\ cos\frac{A+B}{2}\ sin\frac{A-B}{2}}{2\ sin\frac{A+B}{2}\ cos\frac{A-B}{2}} = \frac{K-1}{K+1}$
$\Rightarrow cot\frac{A+B}{2}\tan\frac{A-B}{2} = \frac{K-1}{K+1}$
$\Rightarrow tan\frac{1}{2}(A-B) = \frac{K-1}{K+1}\ tan\frac{1}{2}(A+B)$

Q-5: If $A+B+C=\pi$, Prove that $sin2A + sin2B + sin2C = 4sinAsinBsinC$.
Proof:
L.H.S.= $sin2A + sin2B + sin2C$
$= (sin2A + sin2B) + sin2C$
$= 2\sin(A+B)\cos(A-B) + 2sinC\ cosC$

$= 2sinC\ cos(A-B) + 2sinC\ cosC$ $[\because A+B=\pi-C, \sin(A+B)=\sin(\pi-C)=sinC]$
$= 2sinC\ \{\cos(A-B)+cosC\}$
$= 2sinC(\cos(A-B)-\cos(A+B))$ $[\because cosC=\cos\{\pi-(A+B)\}=-\cos(A+B)]$
$= 2sinC\ (2sinA\ sinB)$
$= 4sinA\ sinB\ sinC =$ RHS

Compound, Multiple and Sub Multiple Angles

Multiple and Sub Multiple Arguments : For an argument (variable) θ usually $2\theta, 3\theta$ etc. are called its multiples and $\theta/2$, $\theta/3$ etc. are called its sub multiples. For arguments θ and $\emptyset$, $\theta+\emptyset$, $\theta-\emptyset$ are called the compound arguments.

Theorem-1

(i) $\boldsymbol{sin2A = 2sinA\ cosA}$

(ii) $\boldsymbol{cos2A = cos^2A - sin^2A = 2cos^2A - 1 = 1 - 2sin^2 A}$

(iii) $\boldsymbol{tan2A = \frac{2tanA}{1-tan^2A};\ A \neq (2n+1)\frac{\pi}{2}}$

Proof:

(i) According to addition theorem, $\sin(A+B) = sinA\ cosB + cosA\ sinB$

Replace the angle B by A, $\sin(A+A) = sinA\ cosA + cosA\ sinA$

Or, $\mathrm{sin2A = 2sinA\ cosA}$

(ii) According to addition theorem, $\cos(A+B) = \mathrm{cosA\ cosB - sinA\ sinB}$

Replace the angle B by A, $\cos(A+A) = \mathrm{cosA\ cosA - sinA\ sinA}$

Or, $\cos 2A = cos^2A - sin^2A$

Again, by using the identity, $sin^2A + cos^2A = 1$ in above formula,

Or, $\cos 2A = cos^2A - (1-cos^2A) = 2cos^2A - 1$

and, $\cos 2A = (1-sin^2A) - sin^2A = 1 - 2sin^2A$

Again, by using the identity, $sin^2A + cos^2A = 1$

(iii) According to addition theorem, $tan(A+B) = \frac{tanA + tanB}{1 - tanA\ tanB}$

Replace the angle B by A, $tan(A+A) = \frac{tanA + tanA}{1 - tanA\ tanA}$

Or, $tan\ 2A = \frac{2\tan A}{1 - tan^2A}$

Theorem-2

(i) $\boldsymbol{sin3A = 3sinA - 4sin^3A}$

(ii) $\boldsymbol{\cos 3A = 4cos^3A - 3cosA}$

(iii)

Proof:

(i) $\sin 3A = \sin(2A+A) = sin2A\ cosA + cos2A\ sinA$
$= (2sinA\ cosA\ )cosA + (1-2sin^2A)sinA$
$= 2sinA\ cos^2A + sinA - 2sin^3A$
$= 2sinA(1-sin^2A) + sinA - 2sin^3A$
$= 3sinA - 4sin^3A$

(ii) $\cos 3A = \cos(3A+A) = \cos 2A \cos A - \sin 2A \sin A$

$$= (2cos^2A - 1)\ \cos A - (2\sin A\cos A)\sin A$$
$$= 2cos^3A - \cos A - 2\ sin^2A\cos A$$
$$= 2cos^3A - \cos A - 2\ (1 - cos^2A)\cos A$$
$$= 2cos^3A - \cos A - 2\ \cos A + 2\ cos^3A$$
$$= 4\ cos^3A - 3\cos A$$

(iii) $\tan 3A = \tan(2A + A) = \frac{\tan 2A + \tan A}{1-\tan 2A\tan A}$

$$= \frac{\left(\frac{2\tan A}{1-\ tan^2A}\right) + \tan A}{1 - \left(\frac{2\tan A}{1-\ tan^2A}\right)\tan A}$$

$$= \frac{3\tan A - tan^3A}{1 - 3\ tan^2A}$$

Note : Replace A by $A/2$, in Theorem-1, the followings can be proved.

(i) $\boldsymbol{sinA = 2sin\frac{A}{2}\ cos\frac{A}{2}}$

(ii) $\boldsymbol{cosA = cos^2\frac{A}{2} - sin^2\frac{A}{2} = 2cos^2\frac{A}{2} - 1 = 1 - 2sin^2\frac{A}{2}}$

(iii) $\boldsymbol{tanA = \frac{2tan\frac{A}{2}}{1-tan^2\frac{A}{2}}}$

Note : Replace A by $A/3$, in Theorem-1, the followings can be derived.

(iv) $\boldsymbol{sin\theta = 3sin\frac{\theta}{3} - 4sin^3\frac{\theta}{3}}$

(v) $\boldsymbol{cos\theta = 4cos^3\frac{\theta}{3} - 3cos\frac{\theta}{3}}$

(vi) $\boldsymbol{tan\theta = \frac{3tan\frac{\theta}{3} - tan^3\frac{\theta}{3}}{1-3tan^2\frac{\theta}{3}}}$

Some Solved Problems

Q-1: Prove that $\frac{cotA-tanA}{cotA+tanA} = cos2A$.

Proof:

LHS: $\frac{cotA-tanA}{cotA+tanA} = \frac{\frac{cosA}{sinA} - \frac{sinA}{cosA}}{\frac{cosA}{sinA} + \frac{sinA}{cosA}} = \frac{\frac{cos^2A - sin^2A}{sinA\ cosA}}{\frac{cos^2A + sin^2A}{sinA\ cosA}}$

$= \frac{cos^2A - sin^2A}{sinA\ cosA} \times \frac{sinA\ cosA}{cos^2A + sin^2A} = cos2A$ =RHS

Q-2: Prove that $cot\frac{A}{2} = \frac{sinA}{1-cosA}$

Proof:

R.H.S= $\frac{sinA}{1-cosA} = \frac{2sin\frac{A}{2}cos\frac{A}{2}}{2sin^2\frac{A}{2}} = \frac{cos\frac{A}{2}}{sin\frac{A}{2}} = cot\frac{A}{2}$ =LHS

Q- 3: Prove that $cotA - cosec2A = cot2A$

Proof:

L.H.S= $cotA - cosec2A = \frac{cosA}{sinA} - \frac{1}{sin2A}$

$$= \frac{cosA}{sinA} - \frac{1}{2sinA\ cosA} = \frac{2cos^2A-1}{2sinA\ cosA}$$

$$= \frac{cos2A}{sin2A} = cot2A$$

Q- 4: Find the value of $sin20°(3 - 4cos^2 70°)$?

Sol:

$$sin20°(3 - 4cos^2 70°) = sin20°[3 - 4cos^2(90° - 20°)]$$

$$= sin20°(3 - 4sin^2 20°)$$

$$= 3sin20° - 4sin^3 20°$$

$$= \sin(3 \times 20°) = sin60° = \frac{\sqrt{3}}{2}$$

Q- 5: Prove that $cos^6A - sin^6A = cos2A\left(1 - \frac{1}{4}sin^2 2A\right)$

Proof:

LHS $= cos^6A - sin^6A = (cos^2A)^3 - (sin^2A)^3$

$$= (cos^2A - sin^2A)(cos^4A + cos^2A\ sin^2A + sin^4A)$$

$$= cos2A\{(sin^2A)^2 + (cos^2A)^2 + sin^2A\ cos^2A\}$$

$$= cos2A\left\{\begin{matrix}(sin^2A)^2 + (cos^2A)^2 + 2sin^2A\ cos^2A \\ -2sin^2A\ cos^2A\ + sin^2A\ cos^2A\end{matrix}\right\}$$

$$= cos2A\{(cos^2A + sin^2A)^2 - sin^2Acos^2A\}$$

$$= cos2A\{1 - (\sin A \cos A)^2\}$$

$$= cos2A\left\{1 - \left(\frac{1}{2}.2\sin A\cos A\right)^2\right\}$$

$= \cos 2A\left(1 - \frac{1}{4}sin^2 2A\right)$ =RHS

Q- 6: Prove that $sin50° - sin70° + sin10° = 0$

Proof:

LHS $= sin50° - sin70° + sin10°$

$$= \sin(60° - 10°) - \sin(60° + 10°) + sin10°$$

$$= -2cos60°\ sin10° + sin10°$$

$$= -2 \times \frac{1}{2}\ sin10° + sin10°$$

$= -sin10° + sin10° = 0 =$ RHS

Q- 7: Find the value of $2sin67\frac{1}{2}^{\circ}\ cos22\frac{1}{2}^{\circ}$.

Ans:

$2\sin\left(90° - 22\frac{1}{2}^{\circ}\right)\cos 22\frac{1}{2}^{\circ} = 2cos22\frac{1}{2}^{\circ}\ cos22\frac{1}{2}^{\circ}$ $[\because \sin(90° - \theta) = cos\theta]$

$$= 2cos^2\ 22\frac{1}{2}^{\circ}$$

$$= 2cos^2 22\frac{1}{2}^{\circ} - 1 + 1$$

$$= cos2 \times 22\frac{1}{2}° + 1$$

$$= cos45° + 1 = \frac{1}{\sqrt{2}} + 1$$

Q- 8: Prove that $cos\frac{\pi}{16} = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}}$

Proof:

we know that $1 + cos\theta = 2cos^2\frac{\theta}{2} \ldots\ldots\ldots\ldots\ldots (1)$

Put $\theta = \frac{\pi}{4}$, $1 + cos\frac{\pi}{4} = 2cos^2\frac{\pi}{8}$

$$\Rightarrow 1 + \frac{1}{\sqrt{2}} = 2cos^2\frac{\pi}{8}$$

$$\Rightarrow 4cos^2\frac{\pi}{8} = 2\left(1 + \frac{1}{\sqrt{2}}\right) = 2 + \sqrt{2}$$

$$\Rightarrow 2cos\frac{\pi}{8} = \sqrt{2 + \sqrt{2}}$$

put $\theta = \frac{\pi}{8}$ in eqn.(1), we get

$$2cos^2\frac{\pi}{16} = 1 + cos\frac{\pi}{8}$$

$$\Rightarrow 4cos^2\frac{\pi}{16} = 2 + 2\,cos\frac{\pi}{8}$$

$$\Rightarrow \left(2cos\frac{\pi}{16}\right)^2 = 2 + \sqrt{2 + \sqrt{2}}$$

$$\Rightarrow 2cos\frac{\pi}{16} = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$$ (Proved)

INVERSE TRIGONOMETRIC FUNCTIONS

Before starting about inverse trigonometric function let's briefly discuss about what is inverse of any function. Corresponding to every Bijection (one-one and onto), $f: A \to B,$ there exists another bijection. It means if we interchange the domain and range of f and let the new fuction denoted by g which will be given by $g: B \to A$ defined by $g(y) = x$ if and only if $f(x) = y$. So $g: B \to A$ is called as the inverse of $f: A \to B$ denoted by f^{-1}.

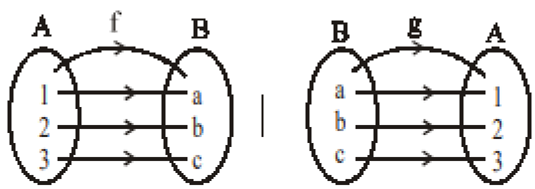


Fig 2.12 Fig 2.13

A function $f: A \to B$ is invertible it means f^{-1}exists if it is one-one and onto. Consider the case of trigonometric functions. In case of sine function: $sin: R \to [-1,1]$. But $\sin 0 = \sin \pi\ = 0$ where $0 \neq \pi$. Thus sine function is not bijective in the domain R. However we see that for $y\ \in [-1,1]$ there exists a unique number x in each of the intervals$\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right], \left[-\,-\frac{\pi}{2}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, ..., such that $\sin x = y$. So if $\sin x = \theta \leftrightarrow x = \sin^{-1}\theta$, and read as "sin inverse of x". The function $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \sec^{-1} x, cosec^{-1}x, \cot^{-1} x$ are called inverse trigonometric function.

Example: We know the $\sin 30^{\circ} = \frac{1}{2}$, when the angle is expressed in degrees and $\sin\frac{\pi}{6} = \frac{1}{2}$, when the angle expressed in radians. It means that, sine of the angle $\pi/6$ radian is $1/2$. The converse statement is, the angle whose sine is ½ is $\pi/6$ radian. Symbolically, it is written as $sin^{-1}\frac{1}{2}$ (whose value is $\pi/6$). The function $\sin^{-1}\theta$ is an number, where as $sin\theta$ is a real number.

Function	Domain(D)	Range(R)
$\sin^{-1} x$	$-1 \leq x \leq 1\ or\ [-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$-1 \leq x \leq 1\ or\ [-1,1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Properties of Inverse Trigonometric Functions

1. Self adjusting property:

(i) $\boldsymbol{\sin^{-1}(sin\theta) = \theta}$

(ii) $\boldsymbol{cos^{-1}(cos\theta) = \theta}$

(iii) $\boldsymbol{tan^{-1}(tan\theta) = \theta}$

Proof: (i) Let $sin\theta = x,\ then\ \theta = sin^{-1}x$.

$\therefore \sin^{-1}(sin\theta) = \sin^{-1} x = \theta$ (Proved)

Similarly, proofs of (ii) & (iii) can be completed..

2. Reciprocal Property:

i. $\mathbf{cosec^{-1}\frac{1}{x} = sin^{-1} x}$

ii. $\mathbf{sec^{-1}\frac{1}{x} = cos^{-1} x}$

iii. $\mathbf{cot^{-1}\frac{1}{x} = tan^{-1} x}$

Proof :

(i) Let $\sin^{-1} x = \theta,\ \Rightarrow x = sin\theta$

Then, $cosec\theta = \frac{1}{\sin\theta} = \frac{1}{x},\ \Rightarrow \theta = cosec^{-1}\frac{1}{x}$

Hence, $= \sin^{-1} x$ and $\theta = \mathrm{cosec}^{-1}\frac{1}{x}$,

Therefore, $\sin^{-1} x = cosec^{-1}\frac{1}{x}$

$\therefore \sin^{-1} x = \mathrm{cosec}^{-1}\frac{1}{x}$ (Proved)

Similarly (ii) and (iii) can be proved.

3. Conversion property:

(i) $\mathbf{sin^{-1} x = cos^{-1}\sqrt{1-x^2} = tan^{-1}\frac{x}{\sqrt{1-x^2}}}$

(ii) $\mathbf{cos^{-1} x = sin^{-1}\sqrt{1-x^2} = tan^{-1}\frac{\sqrt{1-x^2}}{x}}$

Proof:

(i) Let $\theta = \sin^{-1} x$ so that $\sin x = \theta$

Now $cos\theta = \sqrt{1-sin^2\theta} = \sqrt{1-x^2}$

i.e. $\theta = cos^{-1}\sqrt{1-x^2}$

$\therefore$ $\tan\theta = \frac{sin\theta}{cos\theta} = \frac{x}{\sqrt{1-x^2}}$ Or, $\theta = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

Thus, $\theta = \sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$ (Proved)

Similalr (ii) and (iii) can also be proved.

4. Theorem – 1:

(i) $\mathbf{sin^{-1} x + cos^{-1} x = \pi/2}$

(ii) $\mathbf{tan^{-1} x + cot^{-1} x = \pi/2}$

(iii) $\mathbf{sec^{-1} x + cosec^{-1} x = \pi/2}$

Proof:

(i) Let $\sin^{-1} x = \theta$,

$\Rightarrow x = sin\theta = \cos(\pi/2 - \theta)$

$\Rightarrow \cos^{-1} x = {}^{\pi}/_{2} - \theta = {}^{\pi}/_{2} - \sin^{-1} x$

$\Longrightarrow \sin^{-1} x + \cos^{-1} x = \pi/2$ (Proved)

Similarly (ii) and (iii) can also be proved.

1. **Theorem – 2: If $xy < 1$, then $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$**

Proof:

Let $\tan^{-1} x = \theta_1$ and $\tan^{-1} y = \theta_2$ then

$x = \tan\theta_1$ and $y = \tan\theta_2$

Now, $\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1-\tan\theta_1 \tan\theta_2}$

$\Rightarrow \tan(\theta_1 + \theta_2) = \frac{x+y}{1-xy}$

$\Rightarrow (\theta_1 + \theta_2) = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\Rightarrow \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ (Proved)

2. **Theorem – 3: $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$**

Proof:

Let $\tan^{-1} x = \theta_1$ and $\tan^{-1} y = \theta_2$

then $x = \tan\theta_1$ and $y = \tan\theta_2$

Now, $\tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1 \tan\theta_2}$

$\Rightarrow \tan(\theta_1 - \theta_2) = \frac{x-y}{1+xy}$

$\Rightarrow (\theta_1 - \theta_2) = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$\Rightarrow \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ (Proved)

Note:

1. $\boldsymbol{\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)}$

2. $\boldsymbol{2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}\ if\ |x|} < 1$

$\boldsymbol{= \sin^{-1}\frac{2x}{1+x^2}\ if |x| \le 1}$

$\boldsymbol{= \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\ if\ |x| \ge 0}$

3. **Theorem – 4:**

(i) $\boldsymbol{2\sin^{-1} x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]}$

(ii) $\boldsymbol{2\cos^{-1} x = \cos^{-1}\left[2x^2 - 1\right]}$

Proof :

(i) $\sin^{-1} x = \theta$, then $x = \sin\theta$

$\therefore sin2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-sin^2\theta} = 2x\sqrt{1-x^2}$

$\Rightarrow 2\theta = \sin^{-1} 2x\sqrt{1-x^2}$

$\Rightarrow 2\sin^{-1} x = \sin^{-1} 2x\sqrt{1-x^2}$ (Proved)

(ii). Let $\cos^{-1} x = \theta$ then $x = \cos\theta$

$\therefore cos2\theta = 2cos^2\theta - 1$

$\Rightarrow cos2\theta = 2x^2 - 1$

$\Rightarrow 2\theta = \cos^{-1}(2x^2 - 1)$
$\Rightarrow 2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$ (Proved)

4. Theorem-5:

(i) $\mathbf{3\sin^{-1}x = \sin^{-1}(3x - 4x^3)}$
(ii) $\mathbf{3\cos^{-1}x = \cos^{-1}(3x - 4x^3)}$
(iii) $\mathbf{3\tan^{-1}x = \tan^{-1}\frac{3x - x^3}{1 - 3x^2}}$

Proof:
(i) Let $sin^{-1}x = \theta$, $\Longrightarrow x = \sin\theta$
We know that,
$\sin 3\theta = 3sin\theta - 4sin^3\theta$
$\Longrightarrow \sin 3\theta = 3x - 4x^3$
$\Longrightarrow 3\theta = sin^{-1}(3x - 4x^3)$
$\Longrightarrow 3sin^{-1}x = sin^{-1}(3x - 4x^3)$
Similarly, (ii) and (iii) can also be proved.

5. Theorem-6:

(i) $\mathbf{\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)}$
(ii) $\mathbf{\cos^{-1}x + \cos^{-1}y = \cos^{-1}xy - \sqrt{1-x^2}\sqrt{1-y^2}}$
(iii) $\mathbf{\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)}$
(iv) $\mathbf{\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)}$

Proof**:**
(i) Let $\sin^{-1}x = \theta_1$, and $\sin^{-1}y = \theta_2$
Then, $x = \sin\theta_1$ and $y = \sin\theta_2$
$\therefore \sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1 sin\theta_2$
$= sin\theta_1\sqrt{1 - sin^2\theta_2} + \sqrt{1 - sin^2}\theta_1\sin\theta_2$
$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$
$\Rightarrow \theta_1 + \theta_2 = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$
$\Rightarrow \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

Similarly, others can also be proved.

6. Theorem – 7

(i) $\boldsymbol{sin^{-1}(-x) = -sin^{-1}x}$
(ii) $\boldsymbol{cos^{-1}(-x) = \pi - cos^{-1}x}$
(iii) $\boldsymbol{tan^{-1}(-x) = -tan^{-1}x}$

Proof:
(i) Let $-x = \sin\theta$, $\Longrightarrow \theta = sin^{-1}(-x)$ (1)
Since, $-x = \sin\theta$,
$\Longrightarrow x = -\sin\theta = \sin(-\theta)$
$\Longrightarrow -\theta = sin^{-1}x$

$\Longrightarrow \theta = -sin^{-1}x$ (2)

From eqn. (1) and (2), $sin^{-1}(-x) = -sin^{-1}x$ (Proved)

(ii) Let $-x = \cos\theta$, $\Longrightarrow \theta = cos^{-1}(-x)$ (3)

Since, $-x = \cos\theta$,

$\Longrightarrow x = -\cos\theta = \cos(\pi - \theta)$

$\Longrightarrow \pi - \theta = cos^{-1}x$

$\Longrightarrow \theta = \pi - cos^{-1}x$ (4)

From eqn. (3) and (4), $cos^{-1}(-x) = \pi - cos^{-1}x$

(iii) Let $-x = \tan\theta$, $\Longrightarrow \theta = tan^{-1}(-x)$ (5)

Since, $-x = \tan\theta$,

$\Longrightarrow x = -\tan\theta = \tan(-\theta)$

$\Longrightarrow -\theta = tan^{-1}x$

$\Longrightarrow \theta = -tan^{-1}x$ (6)

From eqn. (5) and (6), $tan^{-1}(-x) = -tan^{-1}x$

Some Solved Problems:

Q– 1: Find the value of $\cos\tan^{-1}\cot\cos^{-1}\sqrt{3}/2$

Sol:

$\cos\tan^{-1}\cot\cos^{-1}\sqrt{3}/2$

$= \cos\tan^{-1}\cot\cos^{-1}\cos{}^{\pi}/_{6}$

$= \cos\tan^{-1}\cot{}^{\pi}/_{6}$

$= \cos\tan^{-1}\tan\left({}^{\pi}/_{2} - {}^{\pi}/_{6}\right)$

$= \cos\left({}^{\pi}/_{2} - {}^{\pi}/_{6}\right)$

$= \sin\pi/6 = 1/2$

Q-2: Prove that $tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{3} = \frac{\pi}{4}$

Proof:

LHS= $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$

$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)$ $\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= \tan^{-1}\left(\frac{5/6}{1-1/6}\right) = \tan^{-1}\left(\frac{5/6}{5/6}\right) = \tan^{-1}(1) = \frac{\pi}{4} =$ RHS

Q-3: Prove that $sin^{-1}\frac{4}{5} + sin^{-1}\frac{5}{13} = cos^{-1}\frac{16}{65}$

Proof:

LHS $= sin^{-1}\frac{4}{5} + sin^{-1}\frac{5}{13}$

$= sin^{-1}\left[\frac{4}{5}\sqrt{1 - \left(\frac{5}{13}\right)^2} + \frac{5}{13}\sqrt{1 - \left(\frac{4}{5}\right)^2}\,\right]$ $\left[\begin{array}{c}\because \sin^{-1}x + sin^{-1}y \\ = sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)\end{array}\right]$

$= sin^{-1}\left[\frac{4}{5}\sqrt{1-\frac{25}{169}}+\frac{5}{13}\sqrt{1-\frac{16}{25}}\right]$

$= sin^{-1}\left[\frac{4}{5}\times\frac{12}{13}+\frac{5}{13}\times\frac{3}{5}\right]$

$= sin^{-1}\left(\frac{63}{65}\right) = cos^{-1}\sqrt{1-\left(\frac{63}{65}\right)^2} = cos^{-1}\frac{16}{65} =$ RHS

Q-4: Prove that $2tan^{-1}\frac{1}{3} = tan^{-1}\frac{3}{4}$

Proof:

We know that $2\ tan^{-1}x = tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$\therefore\ 2tan^{-1}\frac{1}{3} = tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right) = tan^{-1}\left(\frac{2/3}{8/9}\right) = tan^{-1}\frac{3}{4}$ =RHS

Q-5: Show that $sin^{-1}\frac{12}{13}+cos^{-1}\frac{4}{5}+tan^{-1}\frac{63}{16} = \pi$

Proof:

LHS= $sin^{-1}\frac{12}{13}+cos^{-1}\frac{4}{5}+tan^{-1}\frac{63}{16}$

$= tan^{-1}\left(\frac{12/13}{\sqrt{1-(12/13)^2}}\right)+tan^{-1}\left(\frac{\sqrt{1-(4/5)^2}}{4/5}\right)+tan^{-1}\frac{63}{16}$ $\left[\begin{array}{l}\because\ sin^{-1}x = tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right), \\ cos^{-1}x = tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\end{array}\right]$

$= tan^{-1}\left(\frac{12}{5}\right)+tan^{-1}\left(\frac{3}{4}\right)+tan^{-1}\frac{63}{16}$

$= tan^{-1}\left(\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right)+tan^{-1}\frac{63}{16}$ $\left[\because tan^{-1}x+tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= tan^{-1}\left(-\frac{63}{16}\right)+tan^{-1}\frac{63}{16}$

$= tan^{-1}0 = \pi =$ RHS

Q – 6: Prove that $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Proof:

L.H.S = $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{2\times\frac{1}{2}}{1-\left({}^{1}/_{2}\right)^2}+\tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{1}{\left(\frac{3}{4}\right)}+\tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\left(\frac{4}{3}\right)\left(\frac{1}{7}\right)}$

$= \tan^{-1} \dfrac{\left({}^{31}/_{21}\right)}{\left({}^{17}/_{21}\right)}$

$= \tan^{-1} \frac{31}{17}$ = R.H.S

Q – 7: Prove that $\cot^{-1} 9 + \operatorname{cosec}^{-1} \sqrt{41}/4 = \pi/4$

Proof:

L.H.S.= $\cot^{-1} 9 + \operatorname{cosec}^{-1} \frac{\sqrt{41}}{4}$

$= \tan^{-1} \dfrac{1}{9} + \tan^{-1} \dfrac{4}{5} \quad (\because \operatorname{cosec}^{-1} \dfrac{\sqrt{41}}{4} = \tan^{-1} \dfrac{4}{5}\ )$

$= \tan^{-1} \dfrac{\frac{1}{9} + \frac{4}{5}}{1 - \left(\frac{1}{9}\right)\left(\frac{4}{5}\right)}$

$= \tan^{-1} \frac{41/45}{41/45} = \tan^{-1} 1 = \pi/4$= R.H.S

Q – 8: If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi,$ then prove that $x^2 + y^2 + z^2 + 2xyz = 1$

Proof:

Given $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$

$\Rightarrow \cos^{-1} x + \cos^{-1} y = \pi - \cos^{-1} z$

$\Rightarrow \cos^{-1} \left(xy - \sqrt{1 - x^2}\ \sqrt{1 - y^2}\right) = \pi - \cos^{-1} z$

$\Rightarrow \left(xy - \sqrt{1 - x^2}\ \sqrt{1 - y^2}\right) = \cos(\pi - \cos^{-1} z)$

$\Rightarrow \left(xy - \sqrt{1 - x^2}\ \sqrt{1 - y^2}\right) = -\cos(\cos^{-1} z)$

$\Rightarrow \left(xy - \sqrt{1 - x^2}\ \sqrt{1 - y^2}\right) = -z$

$\Rightarrow (xy + z) = \left(\sqrt{1 - x^2}\ \sqrt{1 - y^2}\right)$

Squaring both sides,

$\Rightarrow (xy + z)^2 = \left(\sqrt{1 - x^2}\sqrt{1 - y^2}\right)^2$

$\Rightarrow (xy)^2 + z^2 + 2xyz = (1 - x^2)(1 - y^2)$

$\Rightarrow x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

A. Two-dimensional Geometry

Introduction:

Coordinate geometry is a branch of mathematics which deals with the systematic study of geometry by use of algebra. It was first initiated by French Mathematician Rene Descartes (1596-1665) in his book ‘La Geometry’, Published in 1637. Hence it is also known as Cartesian Co-Ordinate Geometry.

Fundamental concept:

In two-dimensional co- ordinate geometry, the position of point on the plane is defined with the help of an ‘order pair of the numbers also known as ‘Co-ordinates’. After determining the Coordinates of the point on a line or curve on the plane, we will find out distance between two points, internal and external division, area of closed figure (Triangle), slope of the lines, consistency of lines ,equations of line and circle using algebra.

Coordinate system:

A system in a plane which involves two mutually perpendicular lines which intersect at the origin and measured with equal units to form a orthogonal system called Cartesian co-ordinate system.
This system is used to specify the location of a point in 2D. (Fig 3.1)

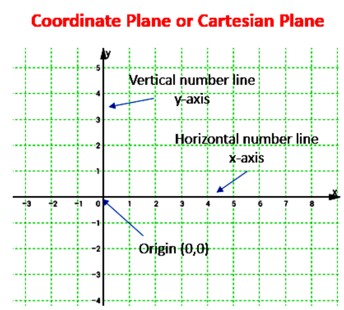


Fig 3.1

Coordinate axes:

The intersecting lines are called coordinates axes.
The horizontal line is called x-axis.
The vertical line is called y- axis
The point of intersection of axes is called the origin.

Origin:

The point where both the axis meet/intersect is called origin and its coordinates are (0,0). From origin towards right through x -axis, ox is measured as +ve units and towards left from origin ox' is –ve. Similarly, from origin towards up through y –axis, oy is +ve and towards down oy' is –ve.

Coordinates:

A pair of numbers which locates the points on the coordinate plane is called its coordinates. It is denoted as an order pair (x , y) .
‘x’ is the distance of a point from the Y-axis is known as **abscissa** or x-coordinate.
‘y’ is the distance of a point from the x-axis is known as **ordinate** or y-coordinate.

Quadrant:

The coordinate axes divide the plane into four equal parts, called quadrants named as
xoy (1st Quadrant) $x > 0, y > 0$
x'o y (2nd Quadrant) $x < 0, y > o$
x'oy'(3rd Quadrant) $x < 0, y < 0$
xoy' (4th quadrant) $x > 0, y < 0$

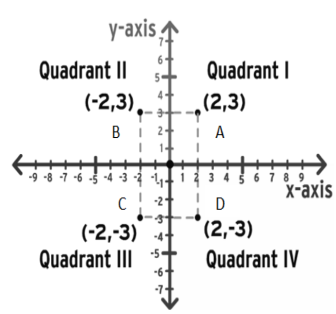


Fig 3.2

Representation of any point (x, y) on the Cartesian plane:
Coordinates of any point on x -axis are (x,0)
Coordinates of any point on y -axis are (0, y)
Example:
Any point A(2,3) is located at 2 unit distance from y -axis measured on ox (positive direction of x-axis right to origin) and 3 units distance from x-axis measured on OY (positive direction of Y axis) .so its lies in 1st quadrant . (figure3.2)
Similarly points B(-2,3),C(-2,-3) and D(2,-3) are located in 2nd ,3rd and 4th quadrant respectively as shown in figure.

Distance formula:
Let P(x_1, y_1) and Q(x_2, y_2) be two given points in the coordinate plane. (Fig 3.3)

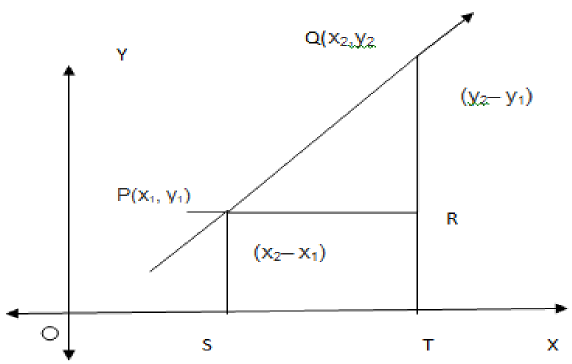


fig 3.3

ΔPQR is a right-angle triangle.
By Pythagoras theorem,

$$PQ^2 = PR^2 + QR^2$$

Or, $$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Or, $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

is the required distance between two given points P(x_1, y_1) and Q(x_2, y_2).
Or, $PQ = \sqrt{(\text{Difference of abscissa})^2 + (\text{Difference of ordinates})^2}$

Distance between a point from the origin:
Distance of a point P(x, y) from the origin O (0, 0) is

$$OP = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

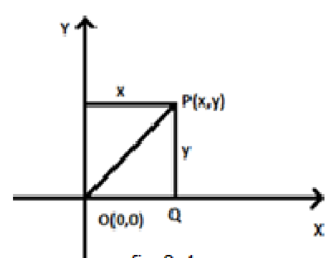


fig 3.4

Some Solved Problems

Q-1: Find the distance between the points P (1, 2) and Q (2, -3).
Sol:

The distance between the points P (1, 2) and Q (2, -3) is

$$|PQ|= \sqrt{(2-1)^2+(-3-2)^2} = \sqrt{1+25} = \sqrt{26} \text{ units.}$$

Q-2: If the distance between the points (3, a) and (6, 1) is 5 find 'a'.

Sol: Given the distance between the points (3, a) and (6, 1) = 5

Using distance formula,

$\sqrt{(6-3)^2+(1-a)^2}$ =5

Or, 9+ $(1-a)^2$=25 (Squaring both sides)

Or, $(1-a)^2 = 16$

Or, 1-a = ±4

Or, a=5, -3

Q-3: If O(0, 0), A(1,0), B(1.1) are the vertices of the triangle, what type of triangle is ΔOAB?

Sol:

Given O(0, 0), A(1,0), B(1, 1) are the vertices of the triangle ΔOAB.

Using distance formula,

$|OA|=\sqrt{(1-0)^2+(0-0)^2}$ = 1

$|OB|=\sqrt{(1-0)^2+(1-0)^2}$ =$\sqrt{2}$

$|AB|=\sqrt{(1-1)^2+(1-0)^2}$ =1

Therefore, OA =AB and $|OA|^2$+$|AB^2|$=$|AB|^2$

Hence, ΔOAB is a right-angle isosceles triangle.

Division/Section formula

Internal division

Let A (x_1, y_1) and B(x_2, y_2) be two given points. Suppose P (x, y) is a point on AB which divides the line AB in the ratio m:n internally i.e. $AP : PB = m : n$.

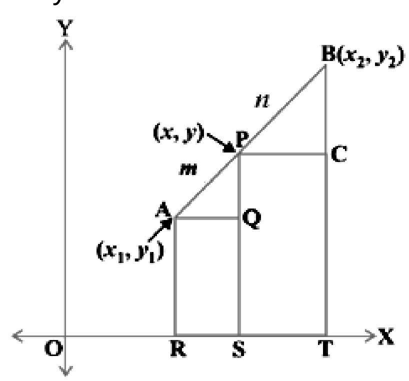


Fig 3.5

From the figure, ΔAPQ and triangle ΔPCB are similar.

Hence, $\frac{AQ}{PC} = \frac{AP}{PB} = \frac{PQ}{BC}$, Or, $\frac{x-x_1}{x_2-x} = \frac{m}{n} = \frac{y-y_1}{y_2-y}$

Now, $\frac{x-x_1}{x_2-x} = \frac{m}{n}$,

Or, $mx_2 - mx = nx - nx_1$

Or, $mx + nx = mx_2 + \ nx_1$

Or, $x = \frac{mx_2 + nx_1}{m+n}$

and $\frac{m}{n} = \frac{y - y_1}{y_2 - y}$

Or, $my_2 - my = ny - ny_1$

Or, $my + ny = my_2 + ny_1$

Or, $y = \frac{my_2 + \ ny_1}{m+n}$

Hence, the coordinates of point P are $\left(\frac{mx_2 + nx_1}{m+n}, \ \frac{my_2 + ny_1}{m+n}\right)$

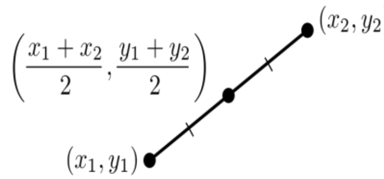


Fig 3.6

Note: Midpoint formula

If R is the midpoint of the line joining P (x_1, y_1) and Q(x_2, y_2),

Then the co-ordinates of R are $\left(\frac{x_1 + x_2,}{2}, \ \frac{y_1 + y_2,}{2}\right)$.

External division:

If A $(x_1 \ , y_1)$ and B(x_2, y_2) be two given points.

Let P(x, y) be any point, which divides the line AB in the ratio m:n externally, then $\frac{AP}{BP} = \frac{m}{n}$

Then the co-ordinates of the point P are $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$

(Similar type of proof as with internal division).

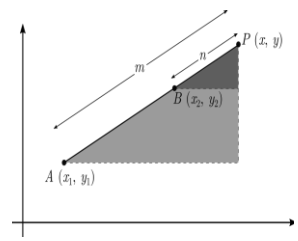


Fig. 3.7

Some Solved Problems

Q-1: Find the coordinates of the point which divides the line joining the points P (1, 2) and Q (3, 4) in the ratio 2:1 internally.

Sol:

Given P (1, 2) and Q (3, 4) be two points.

Let R be the point which divides PQ internally in the ratio 2:1.

Using internal division formula, the co-ordinates of point R are

$$\left(\frac{(2\times3)+(1\times1)}{2+1}, \frac{(2\times4)+(1\times2)}{2+1}\right) = \left(\frac{7}{3}, \frac{10}{3}\right)$$

Q-2: Find the coordinates of the point which divides the line joining the points P(2,3) and Q (–3, 1) in the ratio 3:2 externally.

Sol:

Let R be the point which divides PQ, joining P(2,3) and Q(–3, 1), externally in the ratio 3:1.

Using external division formula,

The co-ordinates of point R are $\left(\frac{(3\times-3)-(2\times2)}{3-2}, \frac{(3\times1)-(2\times3)}{3-2}\right) = (-13, -3)$

Q-3: Find midpoint of the line joining P(2, 3) and Q(4, 5).

Sol:

Let R be the midpoint of the line joining P(2, 3)and Q(4,5).

Using the mid-point formula, the co-ordinates of R are $\left(\frac{2+4}{2}, \frac{3+5}{2}\right) = \ (3, \ 4)$

Q-4: In what ratio does the point $(-1,-1)$ divide the line segment joining the points $(4,4)$ and $(7,7)$?
Sol:
Let the point $C(-1,-1)$ divides the line segment joining the points $A(4,4)$ and $B(7,7)$ in the ratio $k:1$.
Then the co-ordinates of point C are $\left(\frac{7k+4}{k+1}, \frac{7k+4}{k+1}\right)$.
Therefore, $\frac{7k+4}{k+1} = -1$
Or, $\quad 7k + 4 = -k - 1$
Or, $\quad 8k = -5$
Or, $\quad k = -\frac{5}{8}$
Hence, the point C divides AB externally in the ratio 5:8.

Q-5: In what ratio does the x-axis divide the line segment joining the points (2, –3) and (5, 6)?
Sol:
The co-ordinates of the point which divides the line segment joining the points (2, –3) and (5, 6) internally in the ratio K:1 are $\left(\frac{5k+2}{k+1}, \frac{6k-3}{k+1}\right)$.
As, this point lies on x-axis, where y-co-ordinate of every point is zero.
Therefore, $\frac{6k-3}{k+1} = 0$, Or, $6k - 3 = 0$, Or, $k = \frac{1}{2}$
Hence, the required ratio is 1 : 2.

Centroid of a triangle:
Let Δ ABC with vertices are given by
A (x_1, y_1), B(x_2, y_2) and C(x_3, y_3).
Let D, E, F are the midpoints of the side BC, AC and AB respectively.
∴ Coordinates of D, E and F are
$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$, $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$ and $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ respectively.
Let G_1 be a point which divides the median AD internally in the ratio 2:1.

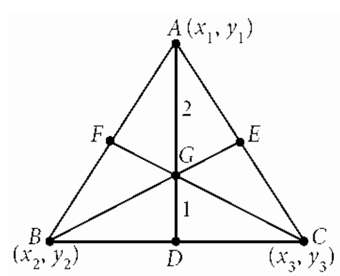


Fig.3.8

So, the co-ordinates of G_1 are $\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+1.x_1}{2+1}, \frac{2\left(\frac{y_2+y_3}{2}\right)+1.y_1}{2+1}\right)$
$$= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$
Similarly, Let G_2 and G_3 be points which divide the median BE and CF internally in the ratio 2:1.
So, the co-ordinates of G_2 and G_3 are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.
Since the co-ordinates of the points are G_1, G_2 and G_3 same i.e. $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$,
So, the points G_1, G_2 and G_3 are not different points but the same point.
Hence, the point having co-ordinates $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ common to AD, BE and CF and divides in the ratio 2:1., which is known as the centroid.

Some Solved Problems

Q-1: Find the co-ordinates of centroid of the triangle whose vertices are (0, 6), (8, 12) and (8, 0).

Sol:

We know that, the co-ordinates of the centroid of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

Therefore, The co-ordinates of the centroid of the triangle with vertices (0, 6), (8, 12) and (8, 0) are $\left(\frac{0+8+8}{3}, \frac{6+12+0}{3}\right) = \left(\frac{16}{3}, 6\right)$

Q-2: Two vertices of a triangle are (1, 2), (3, 5) and its centroid is at the origin. Find the co-ordinates of the third vertex**.**

Sol:

Let the co-ordinates of the third vertex of the triangle be (x, y).

So, the coordinates of the centroid of a triangle with vertices (1, 2). (3, 5) and (x, y) are given by $\left(\frac{1+3+x}{3}, \frac{2+5+y}{3}\right) = \left(\frac{4+x}{3}, \frac{7+y}{3}\right)$,

Given the centroid is at the origin, (0, 0).

Therefore, $\frac{4+x}{3} = 0$ and $\frac{7+y}{3} = 0$, Or, $x = -4$ and $y = -7$.

Hence, the co-ordinates of the third vertex are (–4, –7).

Area of a triangle

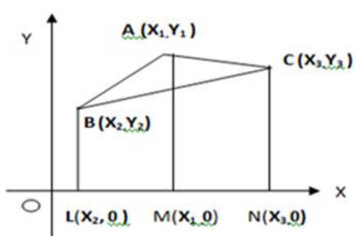


Fig-3.9

Let ABC be a triangle with vertices $A(x_1, y_1), B(x_2, y_2)$ and (x_3, y_3) .

Now, Area of Δ ABC= Area of trapezium ABLM

+ Area of trapezium AMNC

– Area of the trapezium BLNC

We know that, Area of a trapezium= $\frac{1}{2}$ (Sum of parallel sides)(Distance between them)

$\therefore$ Area of ΔABC= $\frac{1}{2}(BL + AM)LM + \frac{1}{2}(AM + CN)MN - \frac{1}{2}(BL + CN)LN$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[x_1(y_2 + y_1 - y_1 - y_3) + x_2(-y_2 - y_1 + y_2 + y_3) + x_3(y_1 + y_3 - y_2 - y_3)]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

This can also be conviniently expressed in the determinant form as

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Notes:

1. If the vertices are taken in anti-clockwise sense, then the area calculated of the triangle will be positive, where as if the points are taken in clockwise, then the area calculated will be negative. But, if the vertices are taken arbitrarily, the area calculated may be positive or negative.
   In case, the area calculated is negative, we consider the numerical /absolute i.e. positive value.
2. To find the area of a polygon, we divide the polygon into some triangles and take the sum of numerical values of area of each triangle.

Co-linearity of three points:

Points A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) are collinear, if they lie on a straight line i.e. area of Δ ABC = 0,

i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Some Solved Problems

Q-1: Find the area of triangle whose vertices are A (4, 4), B (3, –2), and C (–3, 16).

Sol:

Area of the Δ ABC = $\frac{1}{2}\begin{vmatrix} 4 & 4 & 1 \\ 3 & -2 & 1 \\ -3 & 16 & 1 \end{vmatrix}$

$$= \frac{1}{2}[4(-2 - 16) - 4\,(3 + 3) + 1(48 - 6)\,]$$

$$= \frac{1}{2}[-72 - 24 + 42] = \frac{1}{2}(-54) = -27$$

∴ Area of triangle= $\left|-27\right| = 27$ square units

Q- 2: Find the value of 'a', so that area of the triangle having vertices A (0, 0)$_1$ B (1, 0) and C (0, a) is 10 units.

Sol:

Given Area of Δ ABC =10 units

Or, $\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & a & 1 \end{vmatrix} = 10$

Or, 1(a – 0) = 20

Or, a = 20 units

Q-3: Find the value of a so that A (1, 4), B (2, 7) and C (3, a) are Collinear**.**

Sol:

Given A (1, 4), B (2, 7) and C (3, a) are Collinear

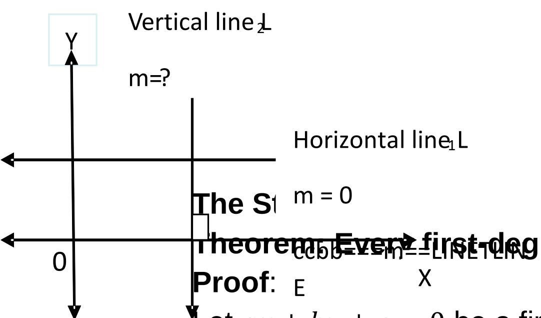
Therefore, Area of Δ ABC = 0

Or, $\frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ 2 & 7 & 1 \\ 3 & a & 1 \end{vmatrix} = 0$

Or, 1 (7 – a) – 4 (2 – 3) + 1 (2a – 21) = 0

Or, 7 – a + 4 + 2a – 21 = 0

Or, a = 10



The S

Theorem: Every first-degree equation in x and y represents a straight line.

Proof:

Let $ax + by + c = 0$ be a first-degree equation in x and y, where a, b and c are constants.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points on the curve represented by $ax + by + c = 0$. Then, $ax_1 + by_1 + c = 0$ and $x_2 + by_2 + c = 0$.

Let R be any point on the line segment joining P and Q. Let R divides PQ in the ratio K:1. Then, the co-ordinates of R are $\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}\right)$.

So, we have, $a\left(\frac{kx_2+x_1}{k+1}\right) + b\left(\frac{ky_2+y_1}{k+1}\right) + c$

$$= \frac{1}{k+1}(akx_2 + ax_1 + kby_2 + by_1 + ck + c)$$

$$= \frac{1}{k+1}[k(ax_2 + by_2 + c) + (ax_1 + by_1 + c)]$$

$$= \frac{1}{k+1}[k(0) + 0] = 0.$$

$\therefore\ R\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}\right)$ lies on the curve represented by $ax + by + c = 0$, and hence every point on the line segment joining P and Q lies on $ax + by + c = 0$.

Hence, $ax + by + c = 0$ represents a straight line.

Slope(Gradient) of a line:

The tangent of the angle made by a line with the positive direction of the x –axis in anticlockwise sense is called slope or gradient of the line.

Generally, the slope of a line is denoted by the letter 'm'.
Hence, $m = tan\theta$, where 'θ' is the angle made by the line with positive direction of x – axis in anticlockwisedirection... (Fig: 3:9)

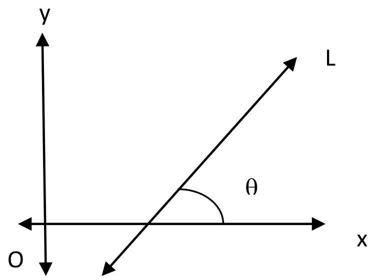


Fig. 3.9

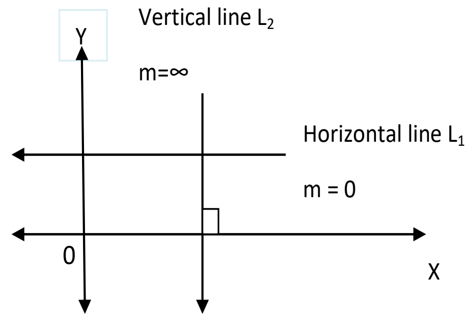


Fig. 3.10

Note: **1.** In (fig 3.10) L_1 is the line parallel to *x*–axis So $\theta = 0^o \Rightarrow m = \tan 0^o = 0$
So, slope of the line parallel to *x*–axis is zero.
Note 2. The line L_2 is perpendicular to *x*-axis or parallel to *y*-axis, so $\theta = 90^o \Rightarrow$ m= $\tan 90^o$, is not defined
So slope of the line parallel to *y*-axis (vertical line) is not defined.
Note 3: Slope of a line equally inclined to both the axes is +1 or –1 , as the line makes with 45° and 135°angle with $x - axis$.

Slope of a line joining two points P (x_1, y_1) and *Q* (x_2, y_2):
Let P (x_1, y_1) and *Q* (x_2, y_2) be two points on a line making an angle θ with positive direction of x-axis.

m=tanθ = $\frac{\text{Perpendicular}}{\text{Base}} = \frac{y_2 - y_1}{x_2 - x_1}$ (fig 3.11)

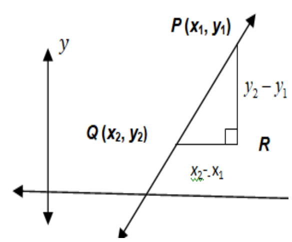


Fig. 3.11

Slope of the line PQ is given by, $m = \frac{y_2 - y_1}{x_2 - x_1}$

Conditions of parallelism and perpendicularity:

1. Two lines L_1 and L_2 are parallel:

Let θ_1 and θ_2 be the angle of inclination of the parallel lines L_1 and L_2.
From fig (3.12), we have $\theta_1 = \theta_2$
$\Rightarrow \tan\theta_1 = \tan\theta_2$
$\Rightarrow m_1 = m_2$
i.e. Two lines are parallel if their slopes are equal.

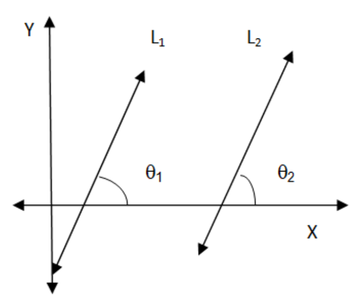


Fig 3.12

2. Two lines are perpendicular to each other

Let θ_1 and θ_2 be the angle of inclination of the perpendicular lines L_1 and L_2.
From Fig. 3.13, we have
$\theta_2 = 90 + \theta_1$

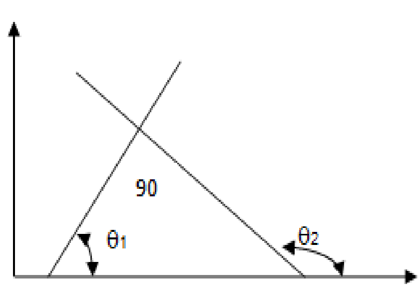


Fig. 3.13

$\Rightarrow \tan\theta_2 = \tan(90 + \theta_1)$

$\Rightarrow m_2 = -\cot\theta_1 = -\frac{1}{\tan\theta_1}$

$\Rightarrow m_2 = \dfrac{-1}{m_1}$

$\Rightarrow m_1 m_2 = -1$

i.e. two lines are perpendicular if their product is equal to –1.

Some Solved Problems

Q- 1: Find Slope of a line joining P (2, 3) and Q (1, 4).

Sol:

Slope of the line joining P (2, 3) and Q (1, 4) = m = $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4-3}{1-2} = \dfrac{1}{-1} = -1$

Q- 2: Find slope of the line perpendicular to a line joining P (1, 2) and Q (3, 5).

Sol:

Slope of the line PQ joining P (1, 2) and Q (3, 5)= $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{5-2}{3-1} = \dfrac{3}{2}$

So, slope of the line perpendicular to PQ = $\dfrac{-1}{3/2} = -2/3$

(Since product of their slopes is -1)

Q- 3: Find slope of the line parallel to the line joining P (1, 4) and Q (2, 6).

Sol:

Slope of the line PQ, joining P (1, 4) and Q (2, 6)= $\dfrac{6-4}{2-1} = \dfrac{2}{1} = 2$

So, slope of the line parallel to PQ = 2. (Since slopes of parallel lines are equal)

Angle between two lines

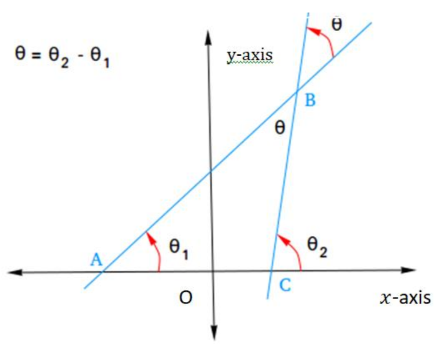


Fig 3.14

Let θ be the angle between two straight lines with slopes m_1 and m_2

i.e $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$,

where θ_1 and θ_2 are the angle of inclinations of two lines.

From Fig. 3.14,

$\theta + \theta_1 = \theta_2$

Or, $\tan\theta = \tan(\theta_2 - \theta_1)$

Or, $\tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_2\tan\theta_1} = \frac{m_2-m_1}{1+m_2m_1}$

The other angle between the lines is given by $\pi - \theta$

So, $\tan(\pi-\theta) = -\tan\theta = -\frac{m_2-m_1}{1+m_2m_1}$

Therefore, the angle (θ) between the lines with slopes m_1 and m_2 is given by

$$\tan\theta = \pm\frac{m_2-m_1}{1+m_1m_2}$$

Note:

The condition of lines to be parallel and perpendicular can also be deduced from the relation

$$\tan\theta = \pm\frac{m_2-m_1}{1+m_1m_2}$$

For parallel lines, $\theta = 0^\circ$,

$\tan\theta = 0$

Or, $\frac{m_2-m_1}{1+m_1m_2} = 0$

Or, $m_1 = m_2$ (Slopes are equal)

and for perpendicular lines, $\theta = 90^\circ$,

Or, $\cot\theta = 0$

Or $\frac{1+m_1m_2}{m_2-m_1} = 0$

Or, $1 + m_1m_2 = 0$

Or, $m_1m_2 = -1$ (Product of their slopes equal to –1)

Some Solved Problems:

Q-1: If $A(-2,1), B(2,3)$ and $C(-2,-4)$ are three points, find the angle between BA and BC.

Sol: Let m_1 and m_2 be the slopes of BA and BC respectively.

$\therefore$ $m_1 = \frac{3-1}{2+2} = \frac{1}{2}$ and $m_1 = \frac{-4-3}{-2-2} = \frac{7}{4}$

Let θ be the angle between BA and BC.

$\therefore$ $\tan\theta = \left|\frac{m_2-m_1}{1+m_1m_2}\right| = \left|\frac{\frac{7}{4}-\frac{1}{2}}{1+\frac{1}{2}\cdot\frac{7}{4}}\right| = \frac{5/4}{15/8} = \frac{2}{3}$

Therefore, the angle between BA and BC $= \theta = tan^{-1}\left(\frac{2}{3}\right)$

Q-2: Determine x so that the line passing through (3, 4) and (x, 5) makes 135°angle with the positive direction of x-axis**.**

Sol: The slope of the line passing through (3, 4) and (x, 5)$= \frac{5-4}{x-3} = \frac{1}{x-3}$

Again, the line makes 135°angle with the positive direction of x-axis,

So its slope$= \tan 135^\circ = -1$.

Therefore, $\frac{1}{x-3} = -1$, Or, $x = 2$

Intercepts of a line on the axes

If a straight-line cuts x-axis at A and the y–axis at B,

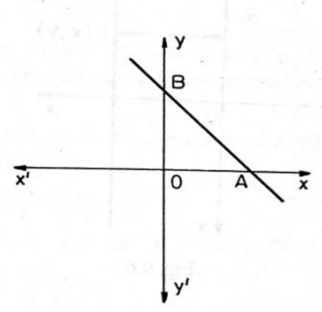


Fig 3.15

then the lengths OA and OB are known as the intercepts of the line x-axis and y-axis respectively.
The intercepts are positive or negative according as the line meets with positive or negative directions of co-ordinate axes.
From Fig. 3.15, OA= x-intercept, OB=y-intercept.
OA is positive or negative according as A lies on OX and OX' respectively. Similarly, OB is positive or negative according as B lies on OY or OY' respectively.

Different forms of equation of a straight line

1. Slope - intercept form:

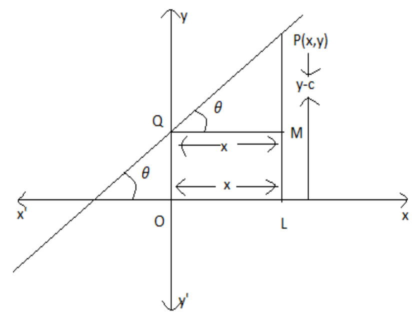


Fig 3.16

Let the given line intersects y-axis at Q
and makes an angle θ with x-axis.
Then $m = \tan\theta$.
Let $P(x, y)$ be any point on the line.
From Fig 3.16,.
Clearly $\angle MQP = \theta, QM = OL = x$ and $PM = PL - ML = PL - OQ$
From triangle PMQ, we have

$$\tan\theta = \frac{PM}{QM} = \frac{y-c}{x}$$

Or, $m = \frac{y-c}{x}$

Or, $y = mx + c$, is the required equation of the line.

Notes:

1. If the line passes through the origin, then $0 = m.0 + c$ or $c = 0$.
   Therefore, the equation of a line passing through the origin is given by $y = mx$.
2. If the line is parallel to x-axis, then m=0.
   Therefore, the equation of a line parallel to x-axis is y=c.
3. If the line is perpendicular to x-axis, then slope of the line 'm' is not defined But $\frac{1}{m} = 0$.
   Therefore, the equation of a line perpendicular to x-axis is $x = \frac{1}{m}y - \frac{c}{m}$. where $\frac{c}{m}$ is the x-intercept. So, $x = 0 - c_1 = c_2$ i.e. $x = constant$ is the equation of line perpendicular to $x -$ axis

Some Solved Problems

Q-1: Find equation of the line which has slope 2 and y intercept 3.
Sol:
Given Slope = m =2 and y-intercept , c = 3
Using slope – intercept form,
Equation of straight line with slope $m = 2$ and y-intercept $c = 3$ is given by

$y = mx + c$

Or, $y = 2x + 3$

Or, $2x - y + 3 = 0$

Q-2: Find the equation of a line with slope 1 and cutting off an intercept 2 units on the negative direction of y-axis**.**

Sol:

Let m be the slope and c be the y-intercept of the required line

Given $m = 1$ and c=–2.

$\therefore$The equation of the line with slope $m = 1$ and y-intercept $c =$ -2 is given by

$y = mx + c$

Or, $y = 1.x + (-2)$

Or, $x - y - 2 = 0$

Q-3: Find the equation of a straight line which cuts off an intercept of 5 units on negative direction of y-axis and makes an angle of $120^\circ$with the positive direction of x-axis**.**

Sol:

Here slope, $m = \tan 120^\circ = \tan(90 + 30^\circ) = -\cot 30^\circ = -\sqrt{3}$

and y-intercept, $c = -5$

Using slope – intercept form $y = mx + c$

Therefore, the equation of the required line is $y = -\sqrt{3}x - 5$ Or, $\sqrt{3}x + y + 5 = 0$

2. One point - slope form:

Let the line with slope m passe through Q(x_1, y_1)

Let P(x,y) be any point on the line .

Then slope of the line is given by m = $\dfrac{y - y_1}{x - x_1}$

Therefore, $y - y_1 = m(x - x_1)$ is the equation of required line.

Some Solved Problems

Q-1: Find equation of the line which passes through (1,2) and slope 2.

Sol:

Using one point-slope form,

Equation of the line passes through (x_1, y_1) =(1,2) and slope m=2 is given by

$y – y_1 = m (x - x_1)$

Or, $y – 2 = 2 (x - 1)$

Or, $2x – y = 0$

Q-2: Determine the equation of line through the point (4, –5) and parallel to x-axis.

Sol:

Since the line is parallel to x-axis, slope , m = 0.

Using point-slope form,

Equation of the line passes through (x_1, y_1) =(4, –5) and slope m=0 is given by

$y – y_1 = m (x - x_1)$

Or, $y + 5 = 0(x - 4)$

Or, $y + 5 = 0$

Q- 3: Find equation of the line which bisects the line segment joining P (1, 2) and Q (3, 4) at right angle.

Sol:

Let R be the mid-point of the line joining P (1, 2) and Q (3, 4).

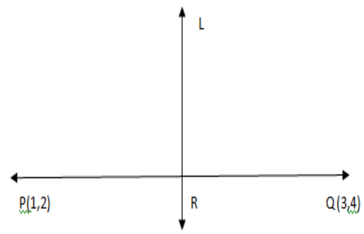


Fig 3.17

So, co-ordinates of R are R $\left(\frac{1+3}{2}, \frac{2+4}{2}\right)$ = R (2, 3)

Now, slope of PQ = m_{PQ} = $\frac{4-2}{3-1} = \frac{2}{2} = 1$

The line LR passes through R (2, 3) and perpendicular to

So Slope = m_{LR} = $-\frac{1}{m_{PQ}}$ = $\frac{-1}{1} = -1$

Equation of the line LR which passes through the point (2, 3) and slope -1 is

$y - y_1 = m(x - x_1)$

Or, $y - 3 = -1(x - 2)$

Or, $y - 3 = -x + 2$

Or, $x + y - 5 = 0$

3. Two-point form:

Let m be the slope of a line passing through two points (x_1, y_1) and (x_2, y_2).

$\therefore$ Slope, $m = \frac{y_2 - y_1}{x_2 - x_1}$

the equation of the required line is

$y - y_1 = m(x - x_1)$ (one point –slope form)

Putting m = $\frac{y_2 - y_1}{x_2 - x_1}$ in above equation, we get

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, is the equation of required line.

Some Solved Problems

Q-1: Find equation of the line which passes through two points P (1, 2) and Q (3, 4).

Sol:

Let m be the slope of the line PQ joining the points P (1, 2) and Q (3, 4).

$\therefore$ Slope= $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{3-1} = 1$

Here, $x_1 = 1, y_1 = 2$ and $x_2 = 3, y_2 = 4$

Using two-point form,

Equation of the required line is

$y - y_1 = m(x - x_1)$

Or, $y - 2 = 1(x - 1)$

Or, $x - y + 1 = 0$

Q-2: Prove that the points (5, 1), (1, –1) and (11, 4) are collinear. Find the equation of the line on which these points lie**.**

Proof:

Let the given points be A(5, 1), B(1, –1) and C(11, 4).

Then the equation of the line passing through A(5, 1) and B(1, –1) is

$$y - 1 = \frac{-1-1}{1-5}(x - 5)$$

Or, $y - 1 = \frac{1}{2}(x - 5)$

Or, $x - 2y - 3 = 0$

Put $x = 11$ and $y = 4$ in the above equation, we get, $11 - 2 \times 4 - 3 = 0$,

Clearly, the point C(11, 4) satisfies the equation $x - 2y - 3 = 0$.

Hence, the given points A, B and C lie on the same straight line and whose equation is $x - 2y - 3 = 0$.

4. Intercept form:

Let AB be a straight line cutting the x-axis and y-axis at A(a, 0) and B(0, b) respectively. (Fig 3.18).

Let x –intercept= $OA = a$ and

Let y-intercept $= OB = b$

Therefore, using two-point form,

The equation of the required straight line passing through A(a, . 0) and $(0, b)$

By two point form, its equation is given by

$$(y - 0) = \frac{b-0}{0-a}(x - a)$$

Or, $y = -\frac{b}{a}(x - a)$

Or, $bx + ay = ab$

Dividing both sides by ab

Or, $\frac{x}{a} + \frac{y}{b} = 1$, is the equation of line in **intercept form**.

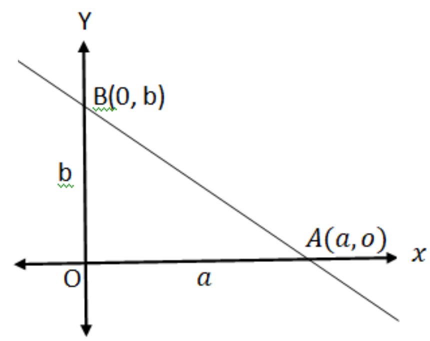


Fig 3.18

Some Solved Problems

Q-1: Find equation of the line which has x-intercept is 2 and y – intercept equal to 3 .

Sol:

Given x-intercept $= a = 2$ and y-intercept $= b = 3$

Using intercept form,

Equation of the straight line is

$$\frac{x}{a} + \frac{y}{b} = 1$$

Or, $\frac{x}{2} + \frac{y}{3} = 1$

Or, $3x + 2y - 6 = 0$

Q-2: Find the equation of the straight line which makes equal intercepts on the axes and passes through the point (2, 3).

Sol:

Let the equation of the line with intercepts a and b is $\frac{x}{a}+\frac{y}{b}=1$.
Since it makes equal intercepts on the co-ordinate axes, then $a=b$
$\therefore$Equation of the line is $\frac{x}{a}+\frac{y}{a}=1$ Or, $x+y=a$
The line $x+y=a$ passes through the point $(2,3)$.
So, $2+3=a$ Or, $a=5$
Hence, the equation of the required line is $\frac{x}{5}+\frac{y}{5}=1$ Or, $x+y=5$.

Q-3: Find the equation of the straight line which passes through the point (3, 4) and the sum of the intercepts on the axes is 14.
Sol:
Let the equation of the line with intercepts a and b be $\frac{x}{a}+\frac{y}{b}=1$ (1)
Given, the line passes through the point (3, 4).
$\therefore \frac{3}{a}+\frac{4}{b}=1$ Or, $3b+4a=ab$ (2)
Also given that, sum of intercepts = 14 i.e. $a+b=14$ (3)
Solving equations (2) and (3), we have,
$$3(14-a)+4a=a(14-a)$$
Or, $a^2-13a+42=0$
Or, $(a-6)(a-7)=0$
Or, $a=6$ and 7
For $a=6$, the value of $b=8$ and for $a=7$, the value of $b=7$.
Putting the values of a and b in equation (1), we get
$$\frac{x}{6}+\frac{y}{8}=1 \text{ and } \frac{x}{7}+\frac{y}{7}=1$$
Or, $4x+3y=24$ and $x+y=7,$ are the equations of the required lines.

5. Normal Form / Perpendicular Form:
Let AB be the line whose equation is to be obtained. From O draw OL perpendicular on AB, then $OL=p$, $\angle AOL=\alpha$.
Let $P(x,y)$ be any point on AB. From P draw PM perpendicular on x-axis.
Then, $\angle A=90°-\alpha$
Therefore, $\angle APM=\alpha$. From ΔAOL, we have, $\frac{OL}{OA}=\cos\alpha$.
Therefore $OL=OA\cos\alpha$.

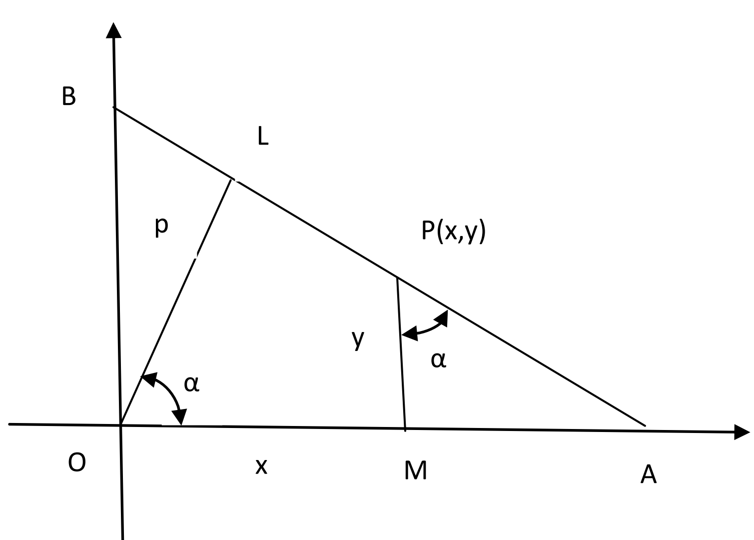


Fig.3.19

Hence, $P = (OM + MA)\cos\alpha = \left(x + \frac{MA}{MP} \times MP\right)\cos\alpha = (x + y.\tan\alpha)\cos\alpha$

$$= \left(x + \frac{\sin\alpha}{\cos\alpha} y\right)\cos\alpha.$$

Therefore, $p = x\cos\alpha + y\sin\alpha.$

is the equation of required line and known as normal or perpendicular form

Some Solved problems

Q-1: Find equation of the line which is at a distance 2 from the origin and the perpendicular from the origin to the line makes an angle of 30° with the positive direction of x –axis**.**

Sol:

The required line is 2 unit distance from the origin,

i.e. the perpendicular distance from the origin to the required line, $p = 2$.

Let α be the angle make by the perpendicular from the origin with positive x –axis.

and given that $\alpha = 30^\circ$.

Using normal form, the equation of the required line is

$x\cos\alpha + y\sin\alpha = p$

Or, $x\cos 30 + y\sin 30 = 2$

Or, $x\,\frac{\sqrt{3}}{2} + y\,\frac{1}{2} = 2$

Or, $\sqrt{3}x + y - 4 = 0$

Transformation of general equation in different standard forms

The general equation of a straight line is $Ax + By + C = 0$ which can be transformed to various standard forms as discussed below.

To transform $\boldsymbol{ax + by + c = 0}$ in the slope intercept form $(\boldsymbol{y = mx + c})$

We have $Ax + By + C = 0$.

Or, $By = -Ax - C$

Or, $y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$, which is of the form $y = mx + c$, where $m = -\frac{A}{B}$, $c = -\frac{C}{B}$.

Thus, for the straight line $Ax + By + C = 0$,

Slope $= m = -\frac{A}{B} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$

and $\text{Intercept on y} - \text{axis} = -\frac{C}{B} = -\frac{\text{Constant term}}{\text{Coefficient of } y}$

Note. To determine the slope of a line by the formula $= -\frac{coefficient\ of\ x}{coefficient\ of\ y}$, transfer all terms in the equation on one side.

To transform $Ax + By + C = 0$ in intercept form $\left(\frac{x}{a} + \frac{y}{b} = 1\right)$

We have $Ax + By + C = 0$

Or, $Ax + By = -C$

Or, $\frac{Ax}{-C} + \frac{By}{-C} = 1$

Or, $\frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1$, which is of the form $\frac{x}{a} + \frac{y}{b} = 1$.

Thus, for the straight line $Ax + By + C = 0$,

Intercept on x-axis$= -\frac{C}{A} = -\frac{\text{Constant term}}{\text{Coefficient of } x}$

Intercept on y-axis$= -\frac{C}{B} = -\frac{\text{Constant term}}{\text{Coefficient of } y}$

Note. As discussed above the intercepts made by a line with the coordinate axes can be determined by reducing its equation to intercept form. We can also use the following method to determine the intercepts on the axes:

For intercepts on x-axis, put $y = 0$ in the equation of the line and find the value of x. Similarly, to find y-intercept, put $x = 0$ in the equation of the line find the value of y.

To transform $Ax + By + C = 0$ in the normal form $(x \cos \alpha + y \sin \alpha = p\,)$.

We have $Ax + By + C = 0$ (1)

Let $x \cos \alpha + y \sin \alpha - p = 0$ (2)

Be the normal form of $Ax + By + C = 0$.

If the equations (1) and (2) represent the same straight line.

Therefore, $\frac{A}{\cos \alpha} = \frac{B}{\sin \alpha} = \frac{C}{-p}$

Or, $\cos \alpha = -\frac{Ap}{C}$ and $\sin \alpha = -\frac{Bp}{C}$ (3)

Or, $\cos^2 \alpha + \sin^2 \alpha = \frac{A^2p^2}{C^2} + \frac{B^2p^2}{C^2}$

Or, $1 = \frac{p^2}{C^2}(A^2 + B^2)$

Or, $p = \pm \frac{C}{\sqrt{A^2+B^2}}$

But p denotes the length of the perpendicular from the origin to the line and is always positive.

Therefore, $p = \frac{C}{\sqrt{A^2+B^2}}$

Putting the value of p in (3), we get $\cos \alpha = -\frac{A}{\sqrt{A^2+B^2}}, \sin \alpha = -\frac{B}{\sqrt{A^2+B^2}}$.

Therefore, equation (2) takes of the form $-\frac{A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y - \frac{C}{\sqrt{A^2+B^2}} = 0$

Or, $-\frac{A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}}$

which is the required normal form of the line $Ax + By + C = 0$.

Note. To transform the general equation to normal form we perform the following steps:

(i) Shift the constant term on the RHS and make it positive

(ii) Divide both sides by $\sqrt{(\text{coefficient of } x)^2 + (\text{coefficient of } y)^2}$.

Some Solved Problems

Q-1 : Transform the equation of the line $\sqrt{3}x + y - 8 = 0$ to

(i) slope intercept form and find its slope and y-intercept.

(ii) intercept form and find intercepts on the coordinate axes.

(iii) normal form and find the inclination of the perpendicular segment from the origin on the line with the axis and its length.

Sol:

(i) We have $\sqrt{3}x + y - 8 = 0$ or $y = -\sqrt{3}x + 8$.

This is the slope intercept form of the given line. Therefore, slope= $-\sqrt{3}$ and y-intercept= 8.

(ii) We have $\sqrt{3}x + y - 8 = 0$ or $\frac{x}{\frac{8}{\sqrt{3}}} + \frac{y}{8} = 1$.

This is the intercept form of the given line. Therefore, x-intercept= $\frac{8}{\sqrt{3}}$, y-intercept= 8.

(iii) We have $\sqrt{3}x + y - 8 = 0$

Or, $\sqrt{3}x + y = 8$

Or, $\frac{\sqrt{3}}{\sqrt{(\sqrt{3})^2 + 1^2}}x + \frac{1}{\sqrt{(\sqrt{3})^2 + 1^2}}y = \frac{8}{\sqrt{(\sqrt{3})^2 + 1^2}}$

Or, $\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 4$.

This is the normal form of the given line. Therefore, $\cos\alpha = \frac{\sqrt{3}}{2}$, $\sin\alpha = \frac{1}{2}$ and $p = 4$.

Since $\sin\alpha$ and $\cos\alpha$ both are positive, therefore α is in first quadrant and is equal to $\alpha = \frac{\pi}{6}$.

Hence $\alpha = \frac{\pi}{6}$ and $p = 4$.

Q-2: Reduce the lines $3x - 4y + 4 = 0$ and $4x - 3y + 12 = 0$ to the normal form and hence determine which line is nearer to the origin.

Sol:

The equation of the given line is $3x - 4y + 4 = 0$

Or, $-3x + 4y = 4$

Or, $-\frac{3x}{\sqrt{(-3)^2+4^2}} + \frac{4y}{\sqrt{(-3)^2+4^2}} = \frac{4}{\sqrt{(-3)^2+4^2}}$

Or, $-\frac{3}{5}x + \frac{4}{5}y = \frac{4}{5}$.

This is the normal form of $3x - 4y + 4 = 0$ and the length of the perpendicular from the origin to it is $p_1 = \frac{4}{5}$.

Again, the equation of second line be $4x - 3y + 12 = 0$

Or, $-4x + 3y = 12$

Or, $-\frac{4x}{\sqrt{(-4)^2+3^2}} + \frac{3y}{\sqrt{(-4)^2+3^2}} = \frac{12}{\sqrt{(-4)^2+3^2}}$

Or, $-\frac{4}{5}x + \frac{3}{5}y = \frac{12}{5}$.

This is the normal form of $4x - 3y + 12 = 0$ and the length of the perpendicular from the origin to it is $p_2 = \frac{12}{5}$.

Clearly, $p_2 > p_1$, therefore the line $3x - 4y + 4 = 0$ is nearer to the origin.

Q-3: Find the equation of a line with slope 2 and the length of the perpendicular from the origin equal to $\sqrt{5}$**.**

Sol:

Let c be the intercept on y-axis.

Then the equation of the line is $y = 2x + c$ (1)

Or, $-2x + y = c$

Or, $-\frac{2}{\sqrt{(-2)^2+1^2}}x + \frac{1}{\sqrt{(-2)^2+1^2}}y = \frac{c}{\sqrt{(-2)^2+1^2}}$ (Dividing both sides by $\sqrt{(-2)^2 + 1^2}$)

Or, $-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y = \frac{c}{\sqrt{5}}$, which is the normal form of (1),

Therefore RHS denotes the length of the perpendicular from the origin. But the length of the perpendicular from the origin is $\sqrt{5}$.

Therefore, $\frac{c}{\sqrt{5}} = \sqrt{5} \Rightarrow c = 5$.

Putting $c = 5$ in (1), we get $y = 2x + 5$,

which is the required equation of the required line.

Q-4: Find equation of the line which passes through P (1, 2) and parallel to the line $x + 2y + 3 = 0$**.**

Sol:

The given line is $x + 2y + 3 = 0$

So, slope, m $= -\frac{coefficient\ of\ x}{coefficient\ of\ y} = -\frac{1}{2}$

Since the required line is parallel to the given line.

Equation of the required line passes through P (1, 2) and $m = -1/2$ is

$y - y_1 = m\ (x - x_1)$

Taking $m = -1/2$

$y - 2 = \frac{-1}{2}\ (x - 1)$

Or, $2y - 4 = -x + 1$

Or, $x + 2y - 5 = 0$

Q-5: Find equation of the line which passes through (2, 3) and perpendicular to the line $3x + 2y + 5 = 0$**.**

Sol:

The equation of given line is $3x + 2y + 5 = 0$

Slope $= m_{given} = \frac{-3}{2}$

Since requires line is perpendicular to the given line,

therefore $m_{required} = \frac{-1}{m_{given}} = \frac{2}{3}$

So, the equation of the required line which passes through (2, 3) and slope 2/3 is

$y - y_1 = m\ (x - x_1)$

Or, $y - 3 = \frac{2}{3}\ (x - 2)$

Or, $3y - 9 = 2x - 4$
Or, $2x - 3y + 5 = 0$

Equation of a line parallel to a given line
Let m be the slope of the line $ax + by + c = 0$.
Then slope, $m = -\frac{a}{b}$ $\left(\text{using } m = -\frac{coefficient\ of\ x}{coefficient\ of\ y}\right)$.
Let c_1 be the y-intercept of the required line.
Therefore, the equation of the required line is
$y = mx + c_1$ (Using slope-intercept form)
Or, $y = -\frac{a}{b}x + c_1$
Or, $ax + by - bc_1 = 0$
Or, $ax + by + \lambda = 0$,   where $\lambda = -bc_1 = \text{constant}$.
Therefore, the equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is a constant.
Note. To write a line parallel to a given line, we keep the expression containing x and y same and simply replace the given constant by a new constant λ. The value of λ can be determined by some given condition.

Equation of a line perpendicular to a given line
Let m_1 be the slope of the given line and m_2 be the slope of a line perpendicular to the given line.
Then $m_1 = -\frac{a}{b}$ and $m_1 m_2 = -1$.   (Using perpendicular condition)
Therefore, $m_2 = -\frac{1}{m_1} = \frac{b}{a}$.
Let c_2 be the y-intercept of the required line. Then its equation is
$y = m_2 x + c_2$
Or, $y = \frac{b}{a}x + c_2$
Or, $bx - ay + ac_2 = 0$
Or, $bx - ay + \lambda = 0$, where $\lambda = ac_2 = \text{constant}$.
Therefore, the equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is a constant
Note. To write a line perpendicular to a given line
(i)   Interchange x and y.
(ii)   If the coefficients of x and y in the given equation are of the same sign, make them of opposite signs and if the coefficients are of opposite signs, make them of the same sign.
(iii)   Replace the given constant by a new constant λ, which is determined by a given condition.

Some Solved problems
Q- 1: Find the equation of the line which is parallel to $3x - 2y + 5 = 0$ and passes through the point $(5, -6)$.
Sol.
The equation of any line parallel to the line $3x - 2y + 5 = 0$ is

$$3x - 2y + \lambda = 0 \qquad (1)$$

The line passes through the point $(5, -6)$.

Thus, $3 \times 5 - 2 \times (-6) + \lambda = 0 \Rightarrow \lambda = -27$.

Putting $\lambda = -27$ in (1), we get $3x - 2y - 27 = 0$ which is the required equation of line.

Q-2: Find the equation of the straight line that passes through the point $(3, 4)$ and perpendicular to the line $3x + 2y + 5 = 0$.

Sol.

The equation of a line perpendicular to $3x + 2y + 5 = 0$ is

$$2x - 3y + \lambda = 0 \qquad (1)$$

The line passes through the point $(3, 4)$.

Thus, $3 \times 2 - 3 \times 4 + \lambda = 0 \Rightarrow \lambda = 6$.

Putting $\lambda = 6$ in (1), we get $2x - 3y + 6 = 0$ which is the required equation of line.

Intersection of two lines:

Let the equations of two lines be

L_1: $A_1x + B_1y + C_1 = 0$ and L_2: $A_2x + B_2y + C_2 = 0$

Slope of $L_1 = -\frac{A_1}{B_1}$ and Slope of $L_2 = -\frac{A_2}{B_2}$

If two line are parallel, i.e. $L_1 \parallel L_2$, then $-\frac{A_1}{B_1} = -\frac{A_2}{B_2}$ Or, $\frac{A_1}{A_2} = \frac{B_1}{B_2}$

If two lines are not parallel to each other they will intersect at a point and solving both the equations, we get the point of intersection.

Concurrency:

Three lines are said to be concurrent if they pass through a common point.

Some Solved Problems

Q-1: Find equation of the line which passes through the point of intersection of two given lines $2x - y - 1 = 0$ and $3x - 4y + 6 = 0$ and parallel to the line $x + y - 2 = 0$.

Sol:

To find the point of intersection of two given lines $2x - y - 1 = 0$ and $3x - 4y - 4 = 0$,

we solve these equations. We get $x = 2$ and $y = 3$

The co-ordinate of point of intersection of two given lines is (2, 3)

Now Slope of the given line $x + y - 2 = 0$ is

$m_{given} = \frac{-A}{B} = \frac{-1}{1} = -1$

Since the required line is parallel to the given line $x + y - 2 = 0$.

Therefore, Slope $m_{req} = m_{given}$ (two lines are parallel)

Or, $m_{req} = -1$

So equation of the line passes through the point (2, 3) with slope -1 is

$$y - 3 = -1\,(x - 2)$$

Or, $x + y - 5 = 0$

Q--2: Find equation of the line which passes through the intersection of the lines $x + 3y + 2 = 0$ and $x - 2y - 4 = 0$ and perpendicular to the line $x + 2y - 1 = 0$.

Sol:

Slope of the given line L_1: $x + 2y - 1 = 0$ = m $_{\text{given}} = -\frac{1}{2}$ (m= $-\frac{A}{B}$)

Slope of the required line(L_2) perpendicular to the line L_1: x + 2y – 1 = 0 is m$_{\text{req}}$ = $-\frac{1}{\frac{-1}{2}}$ =2

To find the intersection point of two lines $x + 3y + 2 = 0$ and $x - 2y - 4 = 0$, we solve these equations and we get $x = \frac{8}{5},\ y = \frac{-6}{5}$,

So, equation of the required line(L_2) passes through the point $\left(\frac{8}{5}, \frac{-6}{5}\right)$ and slope m = 2 is

$$y + \frac{6}{5} = 2\left(x - \frac{8}{5}\right)$$

Or, $2x - y - \frac{22}{5} = 0$

Or, $10x - 5y - 22 = 0$

Perpendicular distance:

(Length of perpendicular from a point P(x_1, y_1) to a line Ax + By + C = 0)

PM is the length of perpendicular from the point P(x_1, y_1) to the line AB which has equation Ax + By + C = 0

$$\left|\overline{PM}\right| = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Note: The length of the perpendicular from the origin to the line $Ax + By + C = 0$ is $\left|\frac{C}{\sqrt{A^2+B^2}}\right|$.

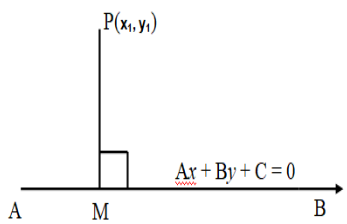


Fig 3.21

Some Solved Problems

Q- 1: Find length of perpendicular from a point (2, 3) to a line $3x - y + 4 = 0$**.**

Sol:

The length of perpendicular from a point (2, 3) to a line $3x - y + 4 = 0$ is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|3\times 2 + (-1)(3) + 4|}{\sqrt{(3)^2 + (-1)^2}} = \frac{|6 - 3 + 4|}{\sqrt{9+1}} = \frac{7}{\sqrt{10}}$$

Q- 2: Find distance between two parallel lines $x + y + 1 = 0$ and $2x + 2y + 3 = 0$.

Sol:

From Fig 3.22,

MN =distance between two parallel lines.

=ON – OM

ON= length of perpendicular from (0, 0) to the line $x + y + 1 = 0$

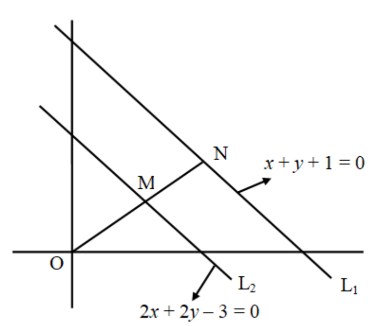


Fig 3.22

is $\dfrac{|1(0)+1(0)+1|}{\sqrt{1+1}}=\dfrac{1}{\sqrt{2}}$

OM= length of perpendicular from (0, 0) to
2*x* + 2*y* - 3 = 0

is $=\dfrac{|2(0)+2(0)-3|}{\sqrt{(2)^2+(2)^2}}=\dfrac{3}{\sqrt{8}}=\dfrac{3}{2\sqrt{2}}$

Therefore, $\mathrm{MN}=\mathrm{ON}-\mathrm{OM}=\frac{1}{\sqrt{2}}-\frac{3}{2\sqrt{2}}=\frac{1}{2\sqrt{2}}$

Alternate method

The distance of the st. line *x* + *y* + 1 = 0 from the origin is given by

$$p_1=\frac{1}{\sqrt{2}}$$

The distance of the st. line 2*x* + 2*y* + 3 = 0 from the origin is given by

$$p_2=\frac{3}{2\sqrt{2}}$$

Since the lines are on the same side of origin
We have,
The distance between the lines = $p_2-p_1=\frac{3}{2\sqrt{2}}-\frac{1}{\sqrt{2}}=\frac{1}{2\sqrt{2}}$

Note: If the lines are on the opposite sides of the origin, then
The distance between the lines = p_2+p_1

Distance between two parallel lines:

If two lines are parallel, then they have the same distance between them throughout. Therefore, to find the distance between two parallel lines choose an arbitrary point on any one of the line and find the length of the perpendicular on the other line. To choose a point on a line give an arbitrary value to *x* or *y*, and find the value of other variable.

Some Solved Problems

Q-1: Find the distance between the parallel lines $3x-4y+9=0$ and $6x-8y-15=0$**.**

Sol:

Putting $y=0$ in $3x-4y+9=0$, we get $x=-3$.
Therefore, $(-3,0)$ is a point on the line $3x-4y+9=0$.
Now, the length of the perpendicular from $(-3,0)$ to $6x-8y-15=0$ is given by

$\left|\frac{6(-3)-8\times 0-15}{\sqrt{6^2+(-8)^2}}\right|=\left|\frac{-33}{\sqrt{100}}\right|=\frac{33}{10}$ units

Q-2: Find the equation of lines parallel to $3x-4y-5=0$ at a unit distance from it.

Sol:

Equation of any line parallel to $3x-4y-5=0$ is

$$3x-4y+\lambda=0 \qquad (1)$$

Putting $x=-1$ in $3x-4y-5=0$, we get $y=-2$.
Therefore, $(-1,-2)$ is a point on $3x-4y-5=0$.
Length of perpendicular from the point $(-1,-2)$ on the line $3x-4y+\lambda=0$ is given by

$\left|\frac{3(-1)-4(-2)+\lambda}{\sqrt{3^2+(-4)^2}}\right| = \left|\frac{5+\lambda}{5}\right|$, which is the distance between two lines.

Given that, Distance between two lines $= 1$

Or, $\left|\frac{5+\lambda}{5}\right| = 1$

Or, $5 + \lambda = \pm 5$

Or, $\lambda = 0$ or -10

Putting the values of λ in equation (1), we get $3x - 4y = 0$ or $3x - 4y - 10 = 0$, which are the equations of required lines.

Circle

Definition:

A circle is the locus of a point which moves on a plane in such a way that its distance from a fixed point is always constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

In the Fig 3.23, $P(x, y)$ is the moving point, C is the centre and CP is the radius.

1. Standard form (Equation of a Circle with given centre and radius)

Let C(α, β) be the centre of the circle and radius of the circle be 'r'.

Let $P(x, y)$ be any point on the circumference of the circle.

Then,

$$CP = r$$

By distance formula,

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = r$$

Or, $(x-\alpha)^2 + (y-\beta)^2 = r^2$

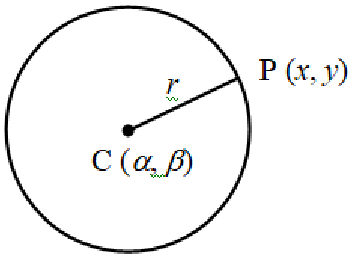


Fig 3.23

Which is the equation of the circle having centre at (α, β) and radius $'r'$, which is known as standard form of equation of a circle.

Note: If the centre of the circle is at origin, $(0, 0)$ and radius is 'r', then the above standard equation of the circle reduces to $x^2 + y^2 = r^2$.

Some Particular Cases:

The standard equation of the circle with centre at $C(\alpha, \beta)$ and radius r, is

$$(x-\alpha)^2 + (y-\beta)^2 = r^2 \quad (1)$$

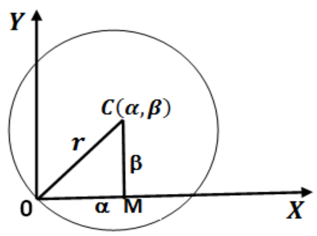


Fig. 3.24

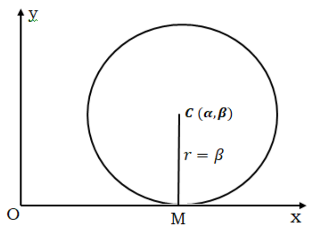


Fig 3.25

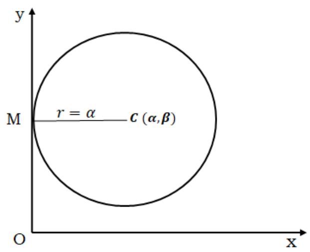


Fig 3.26

(i) **When the circle passes through the origin**

From the Fig 3.24, In right angle triangle ΔOCM,

$OC^2 = OM^2 + CM^2$ i.e. $r^2 = \alpha^2 + \beta^2$

Then eqn (1) becomes,

$$(x-\alpha)^2 + (y-\beta)^2 = \alpha^2 + \beta^2$$

Or, $x^2 + y^2 - 2\alpha x - 2\beta y = 0$

(ii) **When the circle touches x – axis**

In the Fig 3.25, Here, $r = \beta$

Hence, the eqn (1) of the circle becomes,

$$(x-\alpha)^2 + (y-\beta)^2 = \beta^2$$

Or, $x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 = 0$

(iii) **When the circle touches y – axis**

In the Fig 3.26, Here, $r = \alpha$

Hence, the eqn (1) of the circle becomes,

$$(x-\alpha)^2 + (y-\beta)^2 = \alpha^2$$

Or, $x^2 + y^2 - 2\alpha x - 2\beta y + \beta^2 = 0$

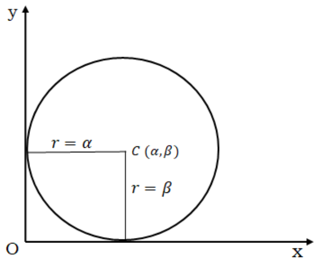


Fig 3.27

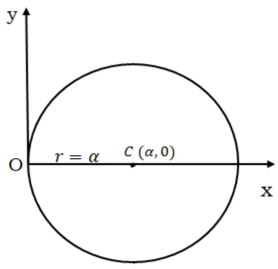


Fig 3.28

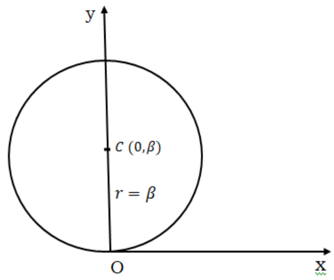


Fig 3.29

(iv) **When the circle touches both the axes**

In the Fig, 3.27, Here, $\alpha = \beta = r$

Hence, the eqn (1) of the circle becomes,

$$(x-r)^2 + (y-r)^2 = r^2$$

Or, $x^2 + y^2 - 2rx - 2ry + r^2 = 0$

(v) **When the circle passes through the origin and centre lies on x- axis**

Here, $\alpha = r$ and $\beta = 0$

Hence, the eqn (1) of the circle becomes,

$$(x-r)^2 + (y-0)^2 = r^2$$

Or, $x^2 + y^2 - 2rx = 0$

(vi) **When the circle passes through the origin and centre lies on y- axis**

Here, $\alpha = 0$ and $\beta = r$

Hence, the eqn (1) of the circle becomes,

$$(x-0)^2 + (y-r)^2 = r^2$$

Or, $x^2 + y^2 - 2ry = 0$

Some Solved Problems

Q- 1: Find equation of the circle which has centre at (2, 3) and radius is 4 .

Sol:

According to the standard form, the equation of circle with centre at (α, β) and radius r is

$$(x - \alpha) + (y - \beta)^2 = r^2$$

∴ Equation of the circle with centre at (2, 3) and radius 4 is,

$$(x - 2)^2 + (y - 3)^2 = (4)^2$$

Or, $x^2 + y^2 - 4x - 6y + 13 = 16$

Or, $x^2 + y^2 - 4x - 6y - 3 = 0$

Q- 2: Find equation the circle which has centre at (1, 4) and passes through a point (2, 6).

Sol:

Given $C(1, 4)$ be the centre and r be the radius of
the circle. The circle passes through the point $P(2, 6)$

$\therefore \quad PC = r$

Or, $\sqrt{(2-1)^2 + (6-4)^2} = r$ (By using distance formula)

Or, $\sqrt{1+4} = r$

Or, $r = \sqrt{5}$

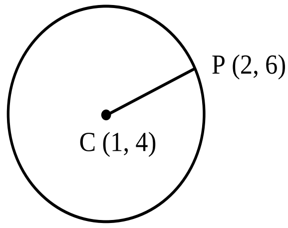


Fig 3.30

By using standard form of the circle,
Equation of the circle with centre at C(1,4) and radius $\sqrt{5}$ is

$$(x - 1)^2 + (y - 4)^2 = \left(\sqrt{5}\right)^2$$

Or, $x^2 + y^2 - 2x - 8y + 1 + 16 = 5$

Or, $x^2 + y^2 - 2x - 8y + 12 = 0$

Q- 3: Find equation of the circle whose centre is at (5, 5) and touches both the axis.

Sol:

The centre of the given circle is at (5, 5).
Since the circle touches both the axes,

$\therefore$ radius, $r = 5$

According to the standard form,

$\therefore$ Equation of the circle with centre at C(5,5) and radius $r = 5$ is

$$(x - 5)^2 + (y - 5)^2 = (5)^2$$

Or, $x^2 + y^2 - 10x - 10y + 25 = 0$

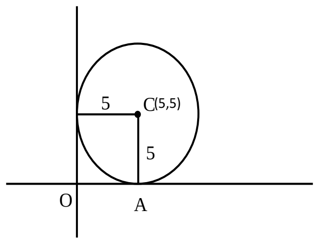


Fig 3.31

Q- 4: If the equation of two diameters of a circle are $x - y = 5$ and $2x + y = 4$, and the radius of the circle is 5, find the equation of the circle.

Sol:

Let the diameters of the circle be AB and LM, whose equations are respectively,

$x - y = 5$ (1)

and $2x + y = 4$ (2)

Since, the point of intersection of any two diameters of a circle is its centre and by solving the equations of two diameters we find the co-ordinates of the centre.

$\therefore$ Solving eqns. (1) and (2), we get $x = 3$ and $y = -2$

Therefore, co-ordinates of the centre are $(3, -2)$ and radius is 5.

Hence, equation of required circle is

$$(x - 3)^2 + (y + 2)^2 = 5^2$$

Or, $x^2 + y^2 - 6x + 4y + 9 + 4 = 25$

Or, $x^2 + y^2 - 6x + 4y - 12 = 0$

Q-5: Find the equation of a circle whose centre lies on positive direction of $y -$ axis at a distance 6 from the origin and whose radius is 4.

Sol:

Given, the centre of the circle lies on positive y-axis at a distance 6 units from origin.

$\therefore$ The centre of the circle lies at the point $C(0, 6)$.

Hence, equation of the circle with centre at C(0, 6) and radius '4' is

$$(x - 0)^2 + (y - 6)^2 = 4^2$$

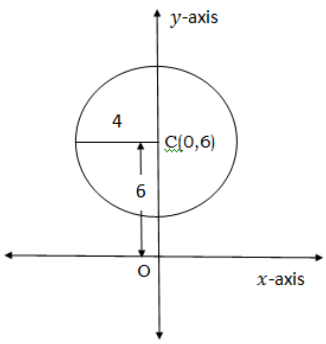


Fig 3.32

Or, $x^2 + y^2 - 12y + 36 = 16$
Or, $x^2 + y^2 - 12y + 20 = 0$

2. General form

Theorem: The equation $\boldsymbol{x^2 + y^2 + 2gx + 2fy + c = 0}$ always represents a circle whose centre is at $\boldsymbol{(-g, -f)}$ and radius is $\boldsymbol{\sqrt{g^2 + f^2 - c}}$.
Proof:
The given equation is $x^2 + y^2 + 2gx + 2fy + c = 0$
Or, $(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$
Or, $(x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$
Or, $\{x - (-g)\}^2 + \{y - (-f)\}^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$
Which is in the standard form (i.e. $(x-\alpha)^2 + (y-\beta)^2 = r^2$) of the circle with centre at (α, β) and radius 'r'.
Hence the given equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle whose centre is $(-g, -f)$ i.e. $\left(-\frac{1}{2}\text{ coeff. of } x, \quad -\frac{1}{2}\text{ coeff. of } y\right)$,
And radius = $\sqrt{g^2 + f^2 - c} = \sqrt{\left(-\frac{1}{2}\text{ coeff. of } x\right)^2 + \left(-\frac{1}{2}\text{ coeff. of } y\right)^2 - \text{constant term}}$

Notes : Characteristics of the general form of equation of circle
The characteristics of general form $x^2 + y^2 + 2gx + 2fy + c = 0$ of a circle are

i. It is quadratic(of second degree) both in x and y.
ii. Coefficient of x^2 = Coefficient of y^2.
iii. It is independent of the term xy, i.e. there is no term containing xy.
iv. Contains three arbitrary constants i.e. g, f and c.

Note : To find the centre and radius of the circle, which is in the form $ax^2 + ay^2 + 2gx + 2fy + c = 0$, where $a \neq 0$**,**
Divide both sides of the equation by coefficient of x^2 or y^2 (i.e. a) to get
$x^2 + y^2 + \frac{2g}{a}x + \frac{2f}{a}y + \frac{c}{a} = 0$,
Which is in the general form of the circle
Hence, the co-ordinates of the centre are $\left(-\frac{1}{2}\text{ coeff. of } x, \quad -\frac{1}{2}\text{ coeff. of } y\right)$

$$= \left(-\frac{1}{2}\frac{2g}{a}, \quad -\frac{1}{2}\frac{2f}{a}\right) = \left(-\frac{g}{a}, \quad -\frac{f}{a}\right)$$

and radius= $\sqrt{\left(-\frac{1}{2}\text{ coeff. of } x\right)^2 + \left(-\frac{1}{2}\text{ coeff. of } y\right)^2 - \text{constant term}}$

$$= \sqrt{\frac{g^2}{a^2} + \frac{f^2}{a^2} - \frac{c}{a}}$$

Example-1:
Let the equation of a circle be $25x^2 + 25y^2 - 30x - 10y - 6 = 0$
To find the centre and radius of the above circle, divide by coefficient of x^2 i.e. 25, as

$$\mathrm{x}^2 + \mathrm{y}^2 - \frac{30}{25}\mathrm{x} - \frac{10}{25}\mathrm{y} - \frac{6}{25} = 0$$

Or, $x^2 + y^2 - \frac{6}{5}x - \frac{2}{5}y - \frac{6}{25} = 0$

Or, $x^2 + y^2 - \frac{6}{5}x - \frac{2}{5}y - \frac{6}{25} = 0$

Or, $x^2 + y^2 + 2\left(-\frac{3}{5}\right)x + 2\left(-\frac{1}{5}\right)y + \left(-\frac{6}{25}\right) = 0$,

which is the general form of circle with centre at $(-g, -f) = \left(\frac{3}{5}, \frac{1}{5}\right)$ and radius

$= \sqrt{\left(-\frac{3}{5}\right)^2 + \left(-\frac{1}{5}\right)^2 - \left(-\frac{6}{25}\right)} = \frac{4}{5}$

Example-2:

Consider the equation of a circle $x(x + y - 6) = y(x - y + 8)$

Or, $x^2 + xy - 6x = xy - y^2 + 8y$

Or, $x^2 + y^2 - 6x - 8y = 0$,

Which, is in the general form of circle.

$\therefore$ Centre $= (-g, -f) = \left(-\frac{1}{2}\text{ coeff. of } x, \quad -\frac{1}{2}\text{ coeff. of } y\right) = \left(-\frac{1}{2}(-6), -\frac{1}{2}(-8)\right) = (3, 4)$

and radius $= \sqrt{g^2 + f^2 - c} = \sqrt{\left(-\frac{1}{2}\text{ coeff. of } x\right)^2 + \left(-\frac{1}{2}\text{ coeff. of } y\right)^2 - \text{constant term}}$

$= \sqrt{\left(-\frac{1}{2}(-6)\right)^2 + \left(-\frac{1}{2}(-8)\right)^2 - 0} = \sqrt{9 + 16 - 0} = 5$

Example-3:

Let the equation of the circle be $x^2 + y^2 + 4x + 6y + 2 = 0$.

Compare this given equation with the general equation of the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$.

Here, $2gx = 4x$, $2fy = 6y$, and $c = 2$

So, $g = 2$, $f = 3$ and $c = 2$

Now, Centre is at (-g, -f) = (-2, -3) and $r = \sqrt{g^2 + f^2 - C} = \sqrt{4 + 9 - 2} = \sqrt{11}$

Some Solved Problems*:*

Q-1 : Determine which of the circles $x^2 + y^2 - 3x + 4y = 0$ and $x^2 + y^2 - 6x + 8y = 0$ is greater.

Sol:

The equations of two given circles are

$C_1: x^2 + y^2 - 3x + 4y = 0$

and $C_2: x^2 + y^2 - 6x + 8y = 0$

In 1st circle C_1, $g = \frac{3}{2}$, $f = 2$, $c = 0$

radius $= r_1 = \sqrt{g^2 + f^2 - c} = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2 - 0} = \sqrt{\frac{9}{4} + 4} = \frac{5}{2}$

Similarly, In 2nd circle C_2, $g = 3$, $f = 4$, $c = 0$

radius $= r_2 = \sqrt{g^2 + f^2 - c} = \sqrt{3^2 + 4^2 - 0} = \sqrt{9 + 16} = 5$

Since, $r_1 < r_2$, So, the 2nd circle $C_2: x^2 + y^2 - 6x + 8y = 0$ is greater.

Q-2: Find the equation of the circle concentric with the circle $x^2 + y^2 - 4x + 6y + 10 = 0$ and having radius 10 units.

Sol:

The coordinates of the centre of the given circle, $x^2 + y^2 - 4x + 6y + 10 = 0$,are

$\left(-\frac{1}{2}\text{ coeff. of } x, \quad -\frac{1}{2}\text{ coeff. of } y\right) = (2, -3)$.

Since the required circle is concentric with the above circle , the centre of the required circle and above given circle are same.

$\therefore$ Centre of the required circle is at $(2,-3)$.

Hence, the equation of the required circle with centre at $(2,-3)$ and radius 10 is

$$(x-2)^2+(y+3)^2=(10)^2$$

Or, $\quad x^2+y^2-4x+6y-87=0$

Q-3: Find the equation of the circle whose centre is at the point (4, 5) and passes through the centre of the circle: $x^2+y^2-6x+4y-12=0$.

Sol:

The co-ordinates of the centre of the circle $x^2+y^2-6x+4y-12=0$ are

$$C_1\left(-\frac{1}{2}\text{ coeff. of }x,\quad -\frac{1}{2}\text{ coeff. of }y\right)=C_1(3,-2).$$

Therefore, the required circle passes through the point $C_1(3,-2)$.

Given, the centre of the required circle is at $C(4,5)$

$\therefore$ radius of the required circle= $CC_1=\sqrt{(4-3)^2+(5+2)^2}=\sqrt{1+49}=\sqrt{50}$

Hence, the equation of the required circle with centre at $C(4,5)$ and radius '$\sqrt{50}$' is

$$(x-4)^2+(y-5)^2=\left(\sqrt{50}\right)^2$$

Or, $\quad x^2+y^2-8x-10y-9=0$

Q-4: Find the equation of the circle concentric with the circle $4x^2+4y^2-24x+16y-9=0$ and having its area equal to 9π sq. units.

Sol:

The equation of given circle is $\quad 4x^2+4y^2-24x+16y-9=0$

Or, $\quad x^2+y^2-6x+4y-\frac{9}{4}=0$

$\therefore \quad$ Centre $\left(-\frac{1}{2}\text{ coeff. of }x,\quad -\frac{1}{2}\text{ coeff. of }y\right)=(3,-2)$.

Since the required circle is concentric with the above circle , the centre of the required circle and above given circle are same.

$\therefore$ Centre of the required circle is $(3,-2)$ and let its radius be 'r'

Again, Given Area of the required circle $=9\pi$

Or, $\quad \pi r^2=9\pi$

Or, $\quad r=3$ units

Therefore, the equation of the required circle with centre at $(3,-2)$ and radius $'3'$ is

$$(x-3)^2+(y+2)^2=(3)^2$$

Or, $\quad x^2+y^2-6x+4y+4=0$

Q-5: Find the equation of the circle concentric with the circle $2x^2+2y^2+8x+12y-25=0$ and having its circumference equal to 6π sq. units.

Sol:

The equation of given circle be $\quad 2x^2+2y^2+8x+12y-25=0$

Or, $\quad x^2+y^2+4x+6y-\frac{25}{2}=0$

$\therefore \quad$ centre $=\left(-\frac{1}{2}\text{ coeff. of }x,\quad -\frac{1}{2}\text{ coeff. of }y\right)=(-2,-3)$.

Since the required circle is concentric with the above circle, the centre of the required circle and above given circle are same.

$\therefore$ Centre of the required circle is $(-2,-3)$ and let its radius be 'r'

Again, Given, circumference of the required circle $= 6\pi$

Or, $2\pi r = 6\pi$

Or, $r = 3$ units

Therefore, the equation of the required circle with centre at $(-2, -3)$ and radius $'3'$ is

$$(x+2)^2 + (y+3)^2 = 3^2$$

Or, $x^2 + y^2 + 4x + 6y + 4 = 0$

Equation of a Circle satisfying certain given conditions

The general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ involves three unknown quantities g, f and c called the arbitrary constants. These three constants can be determined from three equations involving g, f and c. These three equations can be obtained from three independent given conditions. We find the values of g, f and c by solving these three equations, and putting these values in the equation of circle, we get the required equation of circle.

Some Solved Problems:

Q-1: Find equation of the circle passes through the points (0, 0), (1, 0) and (0, 1).

Sol:

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

Since, the circle (1) passes through the points (0, 0), (1, 0) and (0, 1). We have,

$0 + 0 + 0 + 0 + c = 0$, Or. $c = 0$ (2)

$1 + 0 + 2g + 0 + 0 = 0$, Or, $g = -1/2$ (3)

and, $0 + 1 + 0 + 2f + 0 = 0$, Or, $f = -1/2$ (4)

Putting the values of g, f and c in equation (1), we get

$$x^2 + y^2 + 2\left(\frac{-1}{2}\right)x + 2\left(\frac{-1}{2}\right)y + 0 = 0$$

Or, $x^2 + y^2 - x - y = 0$, is the equation of required circle.

Q-2: Find the equation of the circle passes through the points $(0, 2)$, $(3, 0)$ and $(3, 2)$. Also, Find the centre and radius.

Sol:

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

Since, the circle (1) passes through the points (0, 2), (3, 0) and (3, 2) i.e these points lie on the circle (1), we have,

$\therefore$ $0 + 4 + 0 + 4f + c = 0$,

Or, $4f + c = -4$ (2)

$9 + 0 + 6g + 0 + c = 0$

Or, $6g + c = -9$ (3)

and, $9 + 4 + +6g + 4f + c = 0$

Or, $6g + 4f + c = -13$ (4)

On solving equations (2), (3) and (4),

Eqns (2)+(3) : $6g + 4f + 2c = -13$ (5)

Eqns (5)-(4) : $c = 0$

Putting the value of c in (2) and (3), we get

$4f = -4$ Or, $f = -1$

$6g = -9$ Or, $g = -\frac{3}{2}$

Putting the values of $g,\ f$ and c in the general eqn of circle (1), we get

$$x^2 + y^2 + 2\left(-\frac{3}{2}\right)x + 2(-1)y + 0 = 0$$

Or, $x^2 + y^2 - 3x - 2y = 0$, is the equation of required circle.

Now, the centre of the circle= $(-g, -f) = \left(\frac{3}{2},\ 1\right)$

and radius= $\sqrt{g^2 + f^2 - c} = \sqrt{\frac{9}{4} + 1 - 0} = \frac{\sqrt{13}}{2}$

Q-3: Find the equation of the circle which passes through the origin and cuts off intercepts a and b from the positive parts of the axes**.**

Sol:

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

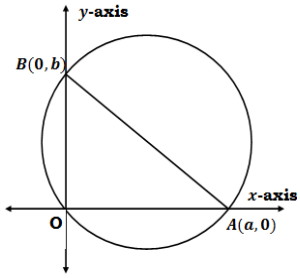


Fig 3.33

Since, the circle passes through the origin and cuts off the intercepts a and b from the positive axes. So, the circle passes through the points $O(0,0)$, $A(a, 0)$ and $B(0, b)$. We have,

$0 + 0 + 0 + 0 + c = 0$, Or, $c = 0$ (2)

$a^2 + 0 + 2ag + 0 + 0 = 0$, Or, $g = -a/2$ (3)

and $0 + b^2 + 0 + 2bf = 0 = 0$, Or, $f = -b/2$ (4)

Putting the values of $g,\ f$ and c in the equation of circle (1), we get

$x^2 + y^2 - ax - by = 0$ is the equation of required circle.

Q-4: Prove that the points (2, –4), (3, –1), (3, –3) and (0, 0) are concyclic.

Sol:

Note : To prove that four given points are concyclic (i.e. four points lie on the circle), we find the equation of the circle passing through any three given points and show that the fourth point lies on it.

Let the equation of the circle passing through the points (0, 0), (2, –4) and (3, –1) be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

Since the point (0, 0) lies on circle (1),we have,

$0 + 0 + 0 + 0 + c = 0$, Or, $c = 0$ (2)

Again, since the point (2, –4) lies on circle (1),we have,

$4 + 16 + 4g - 8f + 0 = 0$,

Or, $4g - 8f = -20$,

Or, $g - 2f = -5$ (3)

Also, since the point (3, –1) lies on circle (1),we have,

$9 + 1 + 6g - 2f + 0 = 0$,

Or, $6g - 2f = -10$,

Or, $3g - f = -5$ (4)

Now, solving equations (3) and (4), we get

$g = -1$ and $f = 2$

Putting the values of g, f and c in the equation of circle (1), we get

$x^2 + y^2 - 2x + 4y = 0$, (5)

is the equation of circle. Now, to check the 4th point (3, –3) lies on the circle (5),we put $x = 3$ and $y = -3$ in eqn (5),

$$9 + 9 - 6 - 12 = 0,$$

Therefore, the point (3, –3) satisfies the equation of circle (5) and lies on the circle.
Hence, the given points are concyclic.

Q-5: Find the equation of circle which passes through (3, –2), (–2, 0) and has its centre on the line $2x - y = 3$.

Sol:

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

Since, the circle (1) passes through the points (3, –2) and (–2, 0) i.e these points lie on the circle (1),we have

$$9 + 4 + 6g - 4f + c = 0$$

Or, $6g - 4f + c = -13$ (2)

and $4 + 0 + 4g + 0 + c = 0$

Or, $4g + c = -4$ (3)

Again, the centre $(-g, -f)$ of circle (1) lies on the line $2x - y = 3$

$\therefore$ $-2g + f = 3$ (4)

Now, solving equations (2), (3) and (4), we get

eqn(2)-eqn(3) : $2g - 4f = -9$ (5)

eqn(4)+eqn(5): $-3f = -6,$ Or, $f = 2$

i.e $-2g + 2 = 3$ Or, $g = -1/2$

i.e $4(-1/2) + c = -4,$ Or, $c = -2$

Putting the values of $g,\ f$ and c in the equation of circle (1), we get

$$x^2 + y^2 + 2\left(-\frac{1}{2}\right)x + 2\,(2)y + (-2) = 0$$

Or, $x^2 + y^2 - x + 4y - 2 = 0$ is the equation of required circle.

Q-6: Find the equation of the circle circumscribing the triangle ΔABC whose vertices are A(1, –5), B(5, 7) and C(–5, 1).

Sol:

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

Since, the circle circumscribing the triangle ΔABC with vertices A(1, –5), B(5, 7) and C(–5, 1), So, the circle (1) passes through the points A(1, –5), B(5, 7) and C(–5, 1).

Therefore,

$$1 + 25 + 2g - 10f + c = 0,$$

Or, $2g - 10f + c = -26$ (2)

$$25 + 49 + 10g + 14f + c = 0$$

Or, $10g + 14f + c = -74$ (3)

and $25 + 1 - 10g + 2f + c = 0$

Or, $-10g + 2f + c = -26$ (4)

On solving equations (2), (3) and (4), we get

eqn(3)-eqn(2): $8g + 24f = -48,$ Or, $g + 3f = -6$ (5)

eqn(3)-eqn(4): $20g + 12f = -48,$ Or, $5g + 3f = -12$ (6)

eqn(5)-eqn(6): $-4g = 6$, Or, $g = -\frac{3}{2}$

i.e $-\frac{3}{2} + 3f = -6$, Or, $f = -\frac{3}{2}$

i.e $2\left(-\frac{3}{2}\right) - 10\left(-\frac{3}{2}\right) + c = -26$, Or, $c = -38$

Putting the values of $g,\ f$ and c in the equation of circle (1),

$$x^2 + y^2 + 2\left(-\frac{3}{2}\right)x + 2\left(-\frac{3}{2}\right)y + (-38) = 0$$

Or, $x^2 + y^2 - 3x - 3y - 38 = 0$ is the equation of required circle.

3. Diameter form (Equation of a circle with given end points of a diameter)

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two end points of diameter AB of a circle. Let $P(x.y)$ be any point on the circle.

Join AP and BP. $\therefore \angle APB = 90^\circ$

($\because$ An angle on a semi-circle is right angle)

Now, slope of $AP = \frac{y-y_1}{x-x_1}$ and slope of $BP = \frac{y-y_2}{x-x_2}$

Since $AP \perp BP$

By condition of perpendicularity, the product of their slopes

Or, $\frac{y-y_1}{x-x_1}\ \frac{y-y_2}{x-x_2} = -1$

Or. $(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$

Or. $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

is the equation of circle with end points of a diameter (x_1, y_1) and (x_2, y_2), which is known as the diameter form of equation of circle.

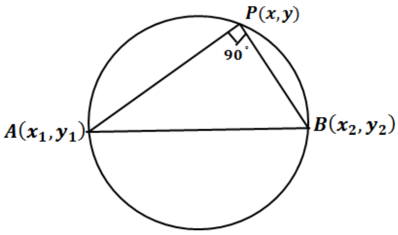


Fig 3.34

Some Solved Problems:

Q- 1 : Find equation of a circle whose end points of a diameter are (1, 2) and (–3, –4).

Sol:

We know that, equation of the circle with end points (x_1, y_1) and (x_2, y_2) of a diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Given, the end points of diameter are (1, 2) and (–3, –4).

Therefore, the equation of the circle is

$$(x - 1)(x + 3) + (y - 2)(y + 4) = 0$$

Or, $x^2 + 2x - 3 + y^2 + 2y - 8 = 0$

Or, $x^2 + y^2 + 2x + 2y - 11 = 0$

Q-2: Find the equation of the circle passing through the origin and making intercepts 4 and 5 on the axes of co-ordinates.

Sol:

Let the intercepts be $OA = 4$ and $OB = 5$.

$\therefore$ The co-ordinates of *A* and *B* are $(4, 0)$ and $(0, 5)$ respectively.

Since $\angle AOB = \frac{\pi}{2}$, therefore AB is the diameter.

According to the diameter form,

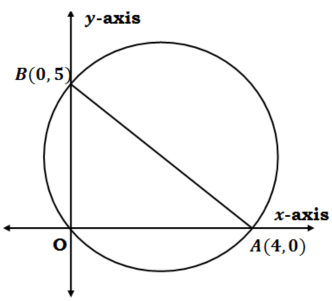


Fig 3.35

A. CO-ORDINATE GEOMETRY IN THREE DIMENSIONS

Three dimensional rectangular co-ordinate system

It is known that the position of point in a plane is determined with reference two mutual perpendicular lines , called axes. Similarly, The position of a point in space is determined by three mutually perpendicular lines or axes.. To specify a point in three dimensional space, we take a fixed point O, called origin and three mutually perpendicular lines through origin, called the X-axis, the Y-axis and the Z-axis. These three axes along with the 'origin' taken together are called the Rectangular co-ordinate system in three dimensions. The three axes are named as X' , YOY' and ZOZ'. The axes in pairs determine three mutually perpendicular planes, XOY, YOZ and ZOX, called the coordinate planes. These planes are known as XY , YZ and ZX plane, respectively. Thus x- axis is perpendicular to YZ- plane , y- axis is perpendicular to ZX- plane and z- axis is perpendicular to XY- plane. The co- ordinate planes divide the space into eight equal compartments called octants. The sign of a point lying on the octant are mentioned below.

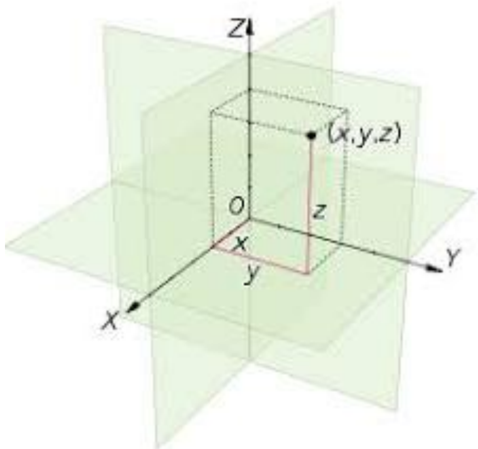


Fig 4.1

Octant	OXYZ	O$X'YZ$	OX$Y'Z$	OXYZ'	O$X'Y'Z$	O$X'YZ'$	OX$Y'Z'$	O$X'Y'Z'$
X	+	-	+	+	-	-	+	-
Y	+	+	-	+	-	+	-	-
Z	+	+	+	-	+	-	-	-

Let P be a point in space. Through P, three planes parallel to co- ordinate planes are drawn to cut the x-axis at A, y-axis at B ,and z-axis at C respectively. Let OA=x, OB=y and OC=z , these numbers x, y, z taken in order are called the co-ordinates of the point P in space and are denoted by P(x, y, z).

SPECIAL CASES

For any point (x, y, z)

i) On XOY – plane , the z co-ordinate is equal to zero i.e P(x , y ,0)

ii) On YOZ – plane , the x co-ordinate is equal to zero i.e P(0 , y ,z)

iii) On ZOX – plane , the y co-ordinate is equal to zero i.e P(x , 0 ,z)

iv) On X- axis the value of y and z are zero (i.e y=z=0) i.e P(x , 0 ,0)
v) On Y- axis the value of x and z are zero (i.e x=z=0) i.e P(0 , y ,0)
vi) On Z- axis the value of x and y are zero (i.e x= y=0) i.e P(0 , 0 ,z)

Note :
The images of the point $P(x\,,y\,,z\,)$ with respect to XY , YZ and ZX plane are $(x\,,y\,,-z\,)$ $(-x\,,y\,,z\,)$ and $(x\,,-y\,,z\,)$ respectively.

Example: Find image of the point $(2\,,-3\,,-4)$ with respect to YZ – plane .
Sol:
the image of the point $(2\,,-3\,,-4)$ with respect to YZ – plane is $(-2\,,-3\,,-4)$

DISTANCE FORMULA
If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two given points, then distance between them is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

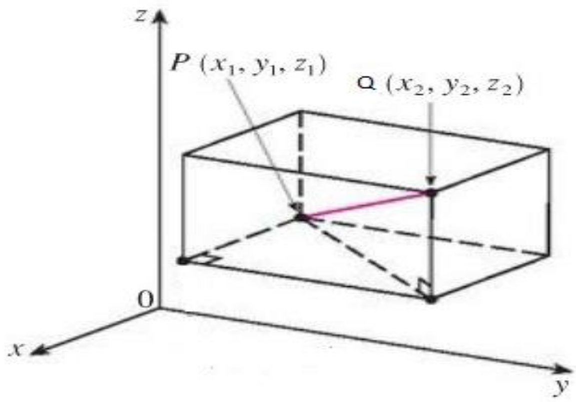


Fig 4.2

Example: Find the value of 'a' if distance between the points $P(0\,,2\,,0\,)$ and $Q(a\,,0\,,4\,)$ is 6.
Sol:
Given that $d = 6,\ P(x_1, y_1, z_1) = (0\,,2\,,0\,)\ \ Q(x_2, y_2, z_2) = (a\,,0\,,4\,)$
Using distance formula,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$d = \sqrt{(a - 0)^2 + (0 - 2)^2 + (4 - 0)^2}$$

Or, $6 = \sqrt{(a)^2 + 4 + 16}$

Or, $36 = a^2 + 20$

Or, $a^2 = 16$

Or, $a = \pm 4$

Note: The perpendicular distance from a point $P(x\,,y\,,z)$ on positive direction

X – axis is $\sqrt{y^2 + z^2}$

Similarly the perpendicular distance from a point $P(x\,,y\,,z)$ on positive direction

Y – axis is $\sqrt{x^2+z^2}$

The perpendicular distance from a point $P(x\,,y\,,z)$ on positive direction Z – axis

is $\sqrt{x^2+y^2}$

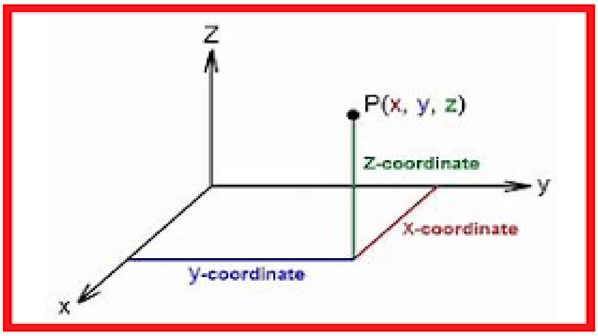


Fig 4.3

DIVISION FORMULA

Internal division formula

Let $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ are two given points . Let $P(x\,,y\,,z)$ be a point which divides the line segment joining the points in the ratio m:n internally. Then co-ordinates of P are

$$x=\frac{mx_2+nx_1}{m+n},\quad y=\frac{my_2+ny_1}{m+n}\ ,\quad \text{and}\quad z=\frac{mz_2+nz_1}{m+n}$$

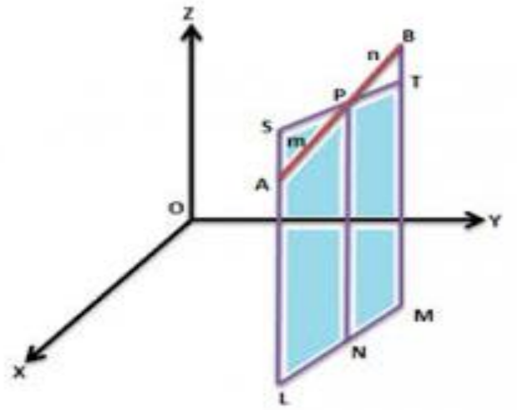


Fig 4.4

External division formula

Let $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ are two given points . Let $P(x\,,y\,,z)$ be a point which divides the line segment in the ratio m:n externally. Then co-ordinates of P are

$$x=\frac{mx_2-nx_1}{m-n},\quad y=\frac{my_2-ny_1}{m-n}\ ,\quad \text{and}\quad z=\frac{mz_2-nz_1}{m-n}$$

Midpoint formula

Let $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ are two given points . Let $P(x\,,y\,,z)$ be the mid point of AB. Then the co-ordinates of P are

$$x=\frac{x_2+x_1}{2},\quad y=\frac{y_2+y_1}{2}\ ,\quad \text{and}\quad z=\frac{z_2+z_1}{2}\qquad \text{(as m/n=1)}$$

DIRECTION COSINES AND DIRECTION RATIOS OF A LINE

Direction cosines:

The cosines of the angles made by a straight line with the positive directions of the co-ordinate axes are called the direction cosines of the line.

Let the line makes an angle $\alpha\,,\beta\ \ and\,\gamma$ with positive direction of X ,Y and Z- axis respectively , then $\cos\alpha\ \,,\cos\beta\ \,,\cos\gamma$ are called direction cosines of that line. In short, it is written as $d.c^s$

Generally , $d.c^s$ of a line are denoted by $l\,,m\ \ and\ n$.

i.e $l = \cos\alpha$, $m = \ \cos\beta \quad n = \cos\gamma$

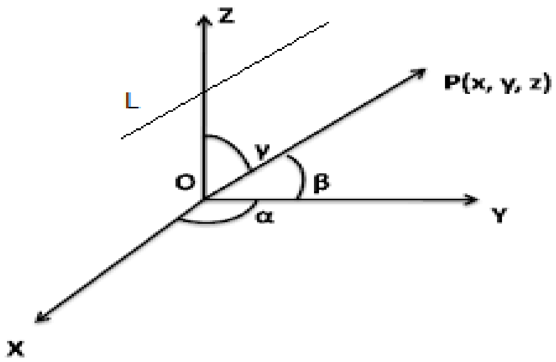


Fig 4.5

Notes :

i. The direction cosines of X – axis are $< 1\,,0\,,0\ >$
   Since x- axis makes an angle 0^0 , 90^0 ,90^0 with co-ordinate axes.
ii. The direction cosines of X – axis in negative direction are $< -1\,,0\,,0\ >$
   Since negative direction of x- axis makes an angle180^0, 90^0, 90^0 with co-ordinate axes.
iii. Similarly, the direction cosines of Y – axis are $< 0\,,1\,,0\ >$
   and the direction cosines of Y – axis in negative direction are $< 0\,,-1\,,0\ >$
iv. The direction cosines of Z – axis are $< 0\,,0\,,1 >$
   and the direction cosines of Z – axis in negative direction are $< 0\,,0\,,-1 >$

Direction ratios

Let a, b and c are three real numbers such that they are proportional to the direction cosine of a line.

I.e $\quad \frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

Then these three numbers are called direction ratios of that line .The direction ratios of a line shortly written as $d.r^s$.

Notes:

i. Direction cosines of two parallel lines are equal but their direction ratios are proportional.
ii. Direction cosines of a line are unique (in respect of magnitude) where as a line may have infinite number of direction ratios .

The sum of squares of the direction cosines of any line is equal to unity

If *l,m,* and *n* are the direction cosines of a line, Then

$$l^2 + m^2 + n^2 = 1$$

Consider a line L having $d.c^s\ l\,, m\ \ and\, n$ and OP be a line segment of length r drawn from origin and parallel to L. Draw perpendicular PA , PB and PC from P on OX , OY and OZ respectively. If co-ordinates of P are (x , y, z)

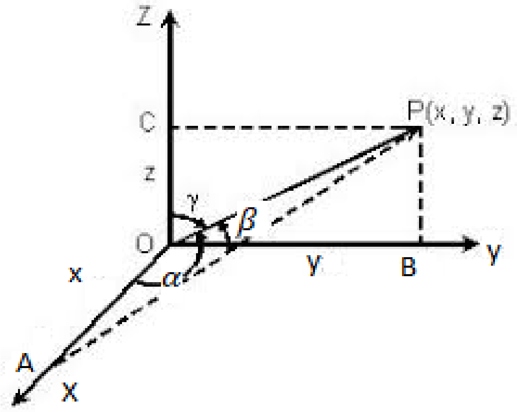


Fig 4.6

Then $\cos\alpha = \frac{x}{r}$ $\quad \cos\beta = \frac{y}{r}$ $\quad \cos\gamma = \frac{z}{r}$

i.e $\quad l = \frac{x}{r}, \qquad m = \frac{y}{r}, \qquad n = \frac{z}{r}$

i.e $\quad x = lr \qquad y = mr \qquad z = nr$

Now, $\quad l^2 + m^2 + n^2$

$= \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2$

$= \frac{x^2+y^2+z^2}{r^2}$

$= \frac{r^2}{r^2} = 1$ (as $x^2+y^2+z^2 = r^2$)

Therefore, $\quad l^2 + m^2 + n^2 = 1$

Direction cosines in terms of direction ratios

We know that $\quad \frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

i.e $\quad \frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\pm\sqrt{a^2+b^2+c^2}}$

i.e $\quad l = \frac{a}{\pm\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\pm\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}$

The two signs correspond to the two angles made by the line with co-ordinate axes.

Example : Find the direction cosines of a line whose direction ratios are 1 ,2,3

Sol:

Given that $a = 1\ , b = 2\ ,\ c = 3$

We known that

$$l = \frac{a}{\pm\sqrt{a^2+b^2+c^2}}\ , m = \frac{b}{\pm\sqrt{a^2+b^2+c^2}}\ , \qquad n = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}$$

i.e $\quad l = \frac{1}{\pm\sqrt{1^2+2^2+3^2}} \qquad m = \frac{2}{\pm\sqrt{1^2+2^2+3^2}} \qquad n = \frac{3}{\pm\sqrt{1^2+2^2+3^2}}$

Or $\quad l = \frac{1}{\pm\sqrt{14}}, \qquad m = \frac{1}{\pm\sqrt{14}} \qquad n = \frac{1}{\pm\sqrt{14}}$

Hence, the direction ratios are $< \frac{1}{\pm\sqrt{14}}, \frac{2}{\pm\sqrt{14}}, \frac{3}{\pm\sqrt{14}} >$

Direction ratios when two points are given.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two given points. Then $d.r.s.$of that line segment PQ are $< x_2 - x_1 , y_2 - y_1 , z_2 - z_1 >$

Projection of a line segment on another line

i. Projection when angle between the lines is given

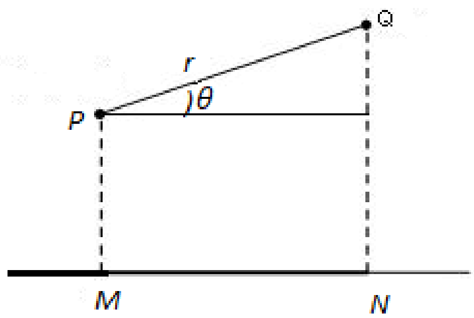


Fig 4.7

Let PQ be the line segment having length r and that makes an angle θ with another line MN , then projection of PQ on MN is

$$MN = PQ \cos\theta = r \cos\theta$$

ii. Projection of a line segment joining two given points

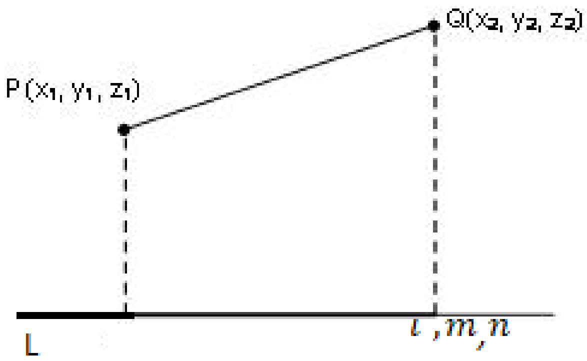


Fig 4.8

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two given points. Then projection of line segment PQ on the line L with direction cosines l , m ,n is

$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

Angle between two lines

Let L_1 and L_2 be two lines having direction cosines are $< l_1, m_1, n_1 >$, $< l_2, m_2, n_2 >$ respectvilly. Let us draw OP and OQ parallel to L_1 and L_2. If θ is the angle between the lines, then

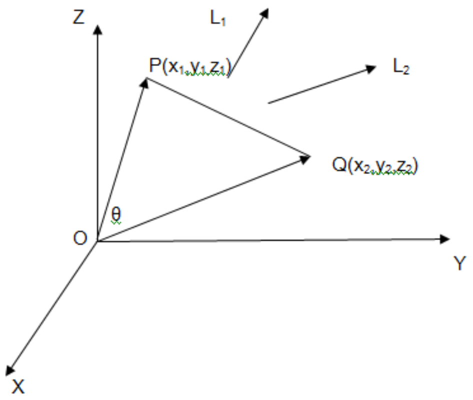


Fig 4.9

In Δ OPQ, $\cos\theta = \frac{OP^2+OQ^2-PQ^2}{2.OP.OQ}$

$$= \frac{(x_1^2+y_1^2+z_1^2)+(x_2^2+y_2^2+z_2^2)-\{(x_2-x_1)^2+(y_2-y_1)^2(z_2-z_1)^2\}}{2.OP.OQ}$$

$$= \frac{x_1x_2+y_1y_2+z_1z_2}{OP.OQ} = \frac{x_1}{OP}\frac{x_2}{OQ} + \frac{y_1}{OP}\frac{y_2}{OQ} + \frac{z_1}{OP}\frac{z_2}{OQ}$$

$= l_1l_2 + m_1m_2 + n_1n_2$ (by property of direction cosines)

$$= \frac{a_1a_2+b_1b_2+c_1c_2}{\pm\sqrt{{a_1}^2+{b_1}^2+{c_1}^2}\sqrt{{a_2}^2+{b_2}^2+{c_2}^2}}$$

Or , $\theta = \cos^{-1}(l_1l_2 + m_1m_2 + n_1n_2)$,

$$\theta = \cos^{-1}\left\{\frac{a_1a_2+b_1b_2+c_1c_2}{\pm\sqrt{{a_1}^2+{b_1}^2+{c_1}^2}\sqrt{{a_2}^2+{b_2}^2+{c_2}^2}}\right\}$$

Where $< a_1, b_1, c_1 >$ and $< a_2, b_2, c_2 >$ are direction ratios of L_1 and L_2 respectively.

Conditions of parallelism and perpendicularity:

i. **Perpendicular lines** :

If L_1 is perpendicular to L_2, then $\cos\theta=0$

Therefore,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

Or, $a_1a_2 + b_1b_2 + c_1c_2 = 0$

ii. **Parallel lines**

Since parallel lines have same direction cosines, it follows from definition of direction ratios that L_1 and L_2 are parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

PLANE

Definition :

A plane is a surface such that the line joining any two points on the surface lies entirely on that surface.

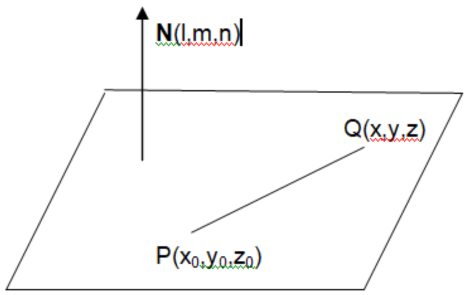


Fig 4.10

Equation of plane passing through a point and whose normal has given directional cosines.

Consider a point P($x_{0,}y_{0,}z_0$) on the plane. Let (l, m, n) be the direction cosines of the normal to the plane. The direction ratios of the line joining the point P($x_{0,}y_{0,}z_0$) and any point Q(x,y,z) on the surface are given as (x-$x_{0,}$), (y-$y_{0,}$), (z-z_0) .Since the direction cosines of the normal to the plane are $l, m, n.$, by condition of perpendicularity

We have

$$l(x - x_0) + m(y - y_0) + n(z - z_0) = 0$$

Which is the equation of plane passing through point ($x_{0,}y_{0,}z_0$) and whose normal has direction cosines $l, m, n.$

Note:

In terms of direction ratios of the normal to the plane, the equation of plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Where a, b, c are direction ratios of the normal to the plane.

Example: Find the equation of plane passing through the point (2,1,3) & normal have direction ratio (1, 1, 1).

Sol:

Given that point (x_0 , y_0 , z_0) = (2, 1 , 3) and (a, b, c) are (1, 1 , 1)

Then equation plane is

a (x-$x_{0,}$) + b (y-$y_{0,}$) + c (z-z_0) = 0

⇒ 1(x-2)+1(y-1)+1(z-3) = 0

⇒ x – 2 + y – 1 + z - 3 = 0

⇒ x + y + z - 6 = 0

So x – y + z = 6 is the required equation of plane.

General form of equation of plane

We have seen the equation of plane passing through point ($x_{0,}y_{0,}z_0$) and whose normal has direction cosines $l, m, n.$ is

$$l(x - x_0) + m(y - y_0) + n(z - z_0) = 0$$

Or $\quad lx + my + nz - (lx_0 + my_0 + nz_0) = 0$

Which is a 1st. degree equation in x,y,z.

We can represent a plane by an equation of 1st. degree in x,y and z.

Therefore, the equation of plane in general form is

$$Ax + By + Cz + D = 0,$$

If the plane passes through the origin, the equation becomes

$$Ax + By + Cz = 0,$$

Let a plane be given by

$$Ax + By + Cz + D = 0$$

If $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are two points on the plane, then

$$Ax_1 + By_1 + Cz_1 + D = 0$$
$$Ax_2 + By_2 + Cz_2 + D = 0$$

Subtraction of the above equations yields

$$A(x_1-x_2) + B(y_1-y_2) + C(z_1-z_2)=0$$

This shows the line with direction ratios (A,B,C) is perpendicular to the line with direction ratios $(x_1-x_2,\ y_1-y_2,\ z_1-z_2)$ i.e to PQ. But P and Q being any points on the plane, we see that the line with d.rs (A,B,C) is perpendicular to every line lying on the plane and hence is normal to the plane.

Thus the direction ratios of the normal to the plane, $Ax + By + Cz + D=0$ can be taken as (A,B,C) i.e coefficients of x, y, z respectively.

Equation of plane under different conditions

I. Equation of plane passing through three given points

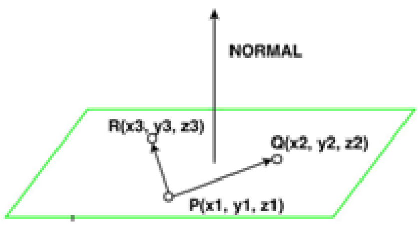


Fig 4.11

Let the plane pass through the points $(x_{1,}y_{1,}z_1)$, $(x_{2,}y_{2,}z_2)$ and $(x_{3,}y_{3,}z_3)$.

Let the equation of the plane be

$$Ax + By + Cz + D = 0$$

Since it passes through the points $(x_{1,}y_{1,}z_1)$, $(x_{2,}y_{2,}z_2)$ and $(x_{3,}y_{3,}z_3)$.

We have

$$Ax_1 + By_1 + Cz_1 + D = 0$$
$$Ax_2 + By_2 + Cz_2 + D = 0$$
$$Ax_3 + By_3 + Cz_3 + D = 0$$

Subtraction of equations yield

$$A(x - x_1) + B(y - y_1) + C(z - z_{1)} = 0$$
$$A(x - x_2) + B(y - y_2) + C(z - z_2) = 0$$
$$A(x - x_3) + B(y - y_3) + C(z - z_3) = 0$$

Eliminating the constants *A, B,C,*

We get

$$\begin{vmatrix} x - x_1 & y - y_{1,} & z - z_1 \\ x_{2,} - x_{1,} & y_{2,} - y_{1,} & z_2 - z_1 \\ x_{3,} - x_{1,} & y_{3,} - y_{1,} & z_3 - z_1 \end{vmatrix} = 0$$

Which is the equation of the plane.

Note: co- planar condition for four points

Four given points (x_1, y_1, z_1), (x_2, y_2, z_2) , (x_3, y_3, z_3) & (x_4, y_4, z_4), are co-planar, if

$$\begin{vmatrix} x_4 - x_1 & y_4 - y_1, & z_4 - z_1 \\ x_{2,} - x_{1,} & y_{2,} - y_{1,} & z_2 - z_1 \\ x_{3,} - x_{1,} & y_{3,} - y_{1,} & z_3 - z_1 \end{vmatrix} = 0$$

Example : Find equation of plane passing through the points(1,3,1),(2,2,1) & (3,2,4).

Sol: We know that equation of plane passing through three given points is

$$\begin{vmatrix} x - x_1 & y - y_{1,} & z - z_1 \\ x_{2,} - x_{1,} & y_{2,} - y_{1,} & z_2 - z_1 \\ x_{3,} - x_{1,} & y_{3,} - y_{1,} & z_3 - z_1 \end{vmatrix} = 0$$

Or, $$\begin{vmatrix} x - 1 & y - 3 & z - 1 \\ 2 - 1 & 2 - 3 & 1 - 1 \\ 3 - 1 & 2 - 3 & 4 - 1 \end{vmatrix} = 0$$

Or, $$\begin{vmatrix} x - 1 & y - 3 & z - 1 \\ 1 & -1 & 0 \\ 2 & -1 & 3 \end{vmatrix} = 0$$

Or $$(x - 1)(-3) - (y - 3)(3) + (z - 1)(-1 + 2) = 0$$

Or $$3x + 3y - z - 11 = 0$$

Which is the required equation of the plane.

II. Equation of a plane parallel to a give plane

Let Ax + By + Cz + D = 0 be the equation of a plane. Then the equation any plane parallel to the above plane is

Ax + By + Cz + K = 0, Where K is constant.

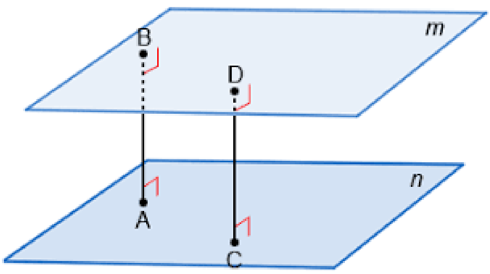


Fig 4.12

Note: Since parallel planes have the same normal, in the equation of parallel plane the coefficients of x, y ,z do not change. Only the constant D will be different.

Example : Find the equation of plane passing through the point (2 , -2 , -1) and parallel to the plane 2x + y - 3z - 2 = 0.

Sol: We know that equation any plane parallel to the plane 2x + y - 3z - 2 = 0 is

2x + y - 3z + K = 0 (1)

Since the plane passes through the point (2,-2,-1) it will satisfy the equation (1),

Thus we have,

$$2 \times 2 - 2 - 3 \times (-1) + K = 0$$

Or, K + 5 =0

Or, K = -5

Putting the above value in eqn(1)

We get

$2x + y - 3z - 5 = 0$

Which represents the required plane

III. Equation of plane passing through intersection of two given planes

Let $A_1X + B_1Y + C_1Z + D_1 = 0$ and $A_2X + B_2Y + C_2Z + D_2 = 0$ be two given planes π_1 and π_2. Then equation of plane passing through intersection of above two plane is given by

$$A_1X + B_1Y + C_1Z + D_1 + K(A_2X + B_2Y + C_2Z + D_2) = 0$$

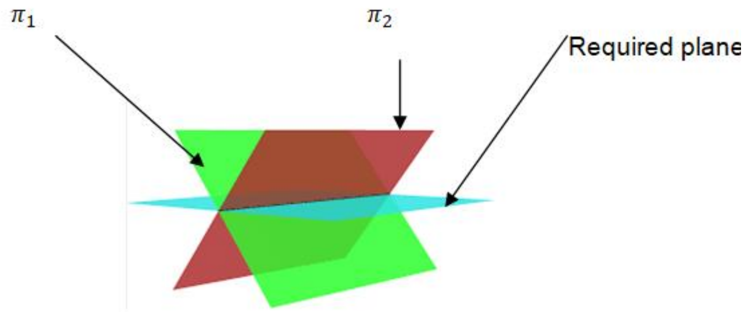


Fig 4.13

Example: Find the equation of plane passing through intersection of planes $2x + 3y - 4z + 1 = 0$, $3x - y + z + 2 = 0$ and passing through the point (3 , 2 , 1).

Sol: We known that equation any plane passing through intersection of the two planes $2x + 3y - 4z + 1 = 0$ and $3x - y + z + 2 = 0$ is

$$(2x + 3y - 4z + 1) + K\,(3x - y + z + 2) = 0 \qquad (1)$$

Since the plane (1) passes through the point (3 , 2 , 1). , we get

$$(2.3 + 3.2 - 4.1 + 1) + K\,(3.3 - 2 + 1 + 2) = 0$$

i.e $\quad 9 + 10K = 0$

Or, $\quad 10K = -9$

Or $\quad K = -9/10$

Putting the value of K in eqn (1), we get the equation of the required plane

$$(2x + 3y - 4z + 1) - 9/10(3x - y + z + 2) = 0$$

i.e $\quad 7x - 39y + 49z + 8 = 0$

Equations of Plane in different forms

I. Equation of plane in intercept form

Let a plane intercept the coordinate axes, X , Y & Z , at a distance a, b and c from origin respectively.

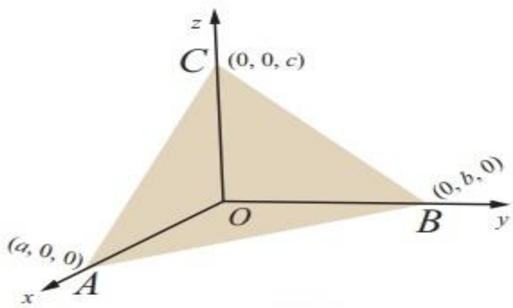


Fig 4.14

Then, the plane passes through the points A(a,0,0),B(0,b,0) and C(0,0,c)

Hence, the equation of the plane is given by

$$\begin{vmatrix} x-a & y-0 & z-0 \\ 0-a & b-0 & 0-0 \\ 0-a & 0-0 & c-0 \end{vmatrix} = 0$$

This on simplification gives

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Which is the equation of plane in **intercept form.**

Example: Find the equation of plane , whose X ,Y ,Z axis intercepts are 1,2,3 respectively.
Sol : Given a = 1, b = 2 & c = 3

So the equation of plane is $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1$

Or, $6x + 3y + 2z - 6 = 0$

II.Equation of plane in Normal form

Let p be the length of the perpendicular ON from the origin on the plane and let *l , m, n* be its direction cosines. Then the coordinates of the foot of the perpendicular N, are (*lp, mp, np*)

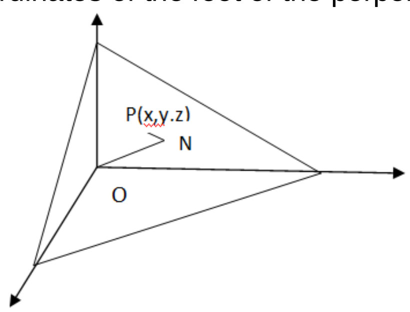


Fig 4.15

If P(x, y, z) be any point on the plane, then the direction cosines of NP are (x-lp, y-np, z-np)
Since ON is perpendicular to the plane, it is also perpendicular to NP
Hence,

$$l(x-lp)+m(y-np)+n(z-np)=0$$

Or $$lx+my+nz=(l^2+m^2+n^2)p$$

Or, $$lx+my+nz=p$$

Is the equation of plane in **Normal form.**

Angle between two intersecting planes

Let us consider two intersecting planes

$$A_1X+B_1Y+C_1Z+D_1=0$$
$$A_2X+B_2Y+C_2Z+D_2=0$$

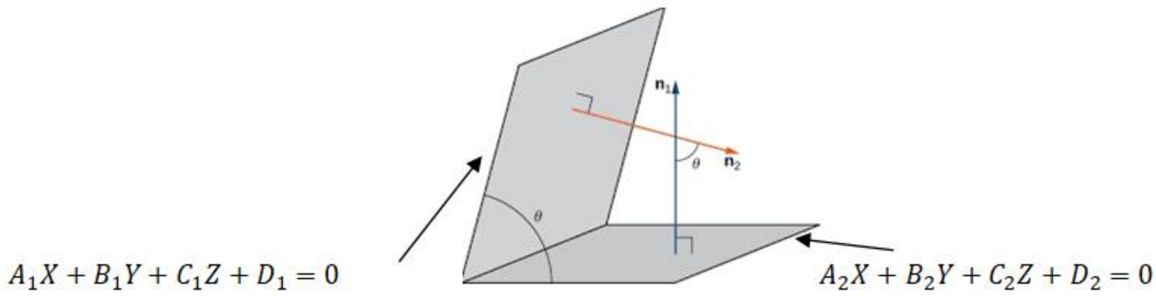


Fig 4.16

Let θ be the angle between the planes. It is obvious that θ also measures an angle between the normals to the planes.
Therefore,

$$\theta = \cos^{-1}\left(\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1{}^2+B_1{}^2+C_1{}^2}\sqrt{A_2{}^2+B_2{}^2+C_2{}^2}}\right)$$

Note: Two planes $A_1X + B_1Y + C_1Z + D_1 = 0$, $A_2X + B_2Y + C_2Z + D_2 = 0$

I. Are **parallel** to each other if their normals are parallel

i.e $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

(This ratio can be conveniently taken as 1 while solving problems)

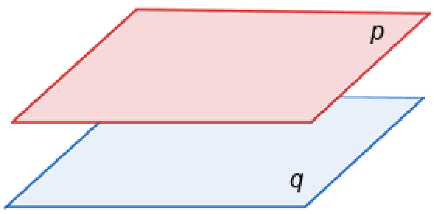


Fig 4.17

II. Are **perpendicular** to each other if their normals are perpendicular to each other

i.e $A_1A_2 + B_1B_2 + C_1C_2 = 0$

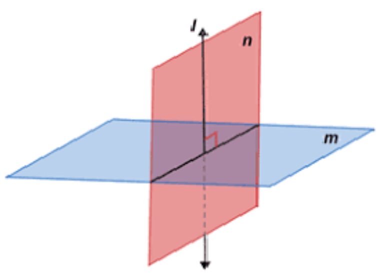


Fig 4.18

III. Are **coincident**

If $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}$

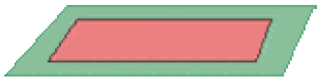


Fig 4.19

Perpendicular distance of a point from the plane

Let P(x_0, y_0, z_0) be a point on the plane, Ax + By + Cz + D = 0. Then perpendicular distance from the point to the plane is

$$p = \frac{Ax_0 + By_0 + Cz_0 + D}{\pm\sqrt{A^2+B^2+C^2}}$$

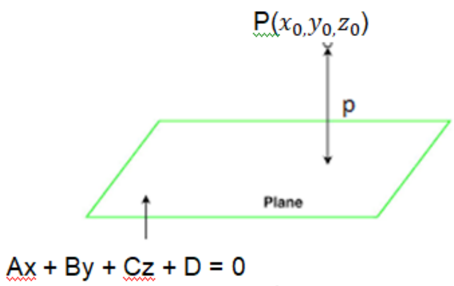


Fig 4.20

Note : The perpendicular distance from origin (0, 0, 0) to the plane Ax + By + Cz + D = 0 is,

$$p = \frac{D}{\pm\sqrt{A^2+B^2+C^2}}$$

SPHERE

A sphere is the locus of a point in space so that its distance from a fixed point is constant The fixed point is called **centre** of the sphere and fixed distance is called **radius** of the sphere. The radius is generally denoted by ‘r’.

Equation of sphere having centre at (a, b , c) and radius r

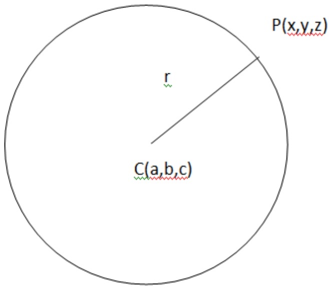


Fig 4.21

By distance formula,

$$CP^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$$

Or,

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Is the required equation of the sphere.

Note: If the centre of the sphere is at origin i.e (0, 0, 0) then its equation becomes

$$x^2+y^2+z^2=r^2$$

Example : Find the equation of sphere having center at $(-2,4,3)$ and passing through the point $(2,1,3)$,

Sol: Given that C $(a, b\ c)=(-2,4,3)$ and point P$(2,1,3)$,

Now radius, $CP=r=\sqrt{(2+2)^2+(1-4)^2+(3-3)^2}$

Or, $\quad r=\sqrt{16+9+0}$

Or, $\quad r=\sqrt{25}$

Or, $\quad r=5$

So equation of sphere is

$$r^2=(x-a)^2+(y-b)^2+(z-c)^2$$

Or, $\quad 5^2=(x+2)^2+(y-4)^2+(z-3)^2$

Or, $\quad (x+2)^2+(y-4)^2+(z-3)^2=25$

Or, $\quad x^2+y^2+z^2+4x-8y-6z+4=0$.

Equation of sphere when end points of diameter are given

Let A (x_1, y_1, z_1) and B (x_2, y_2, z_2) are two end points of a diameter of the sphere. If we consider any point P(x, y, z) on the sphere, then, $\angle APB=90^0$ i.e AP is perpendicular to BP. Since the direction ratios of AP and BP are $(x-x_1, y-y_1, z-z_1)$ and $(x-x_2, y-y_2, z-z_2)$ respectively, by condition of perpendicularity. we have

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

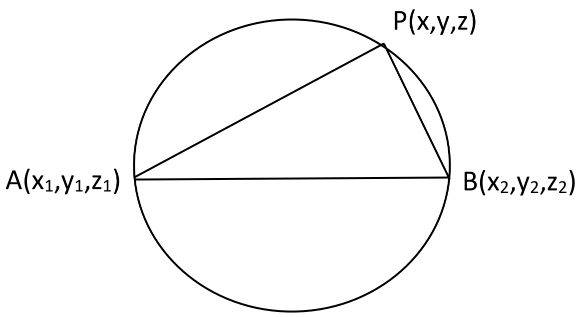


Fig 4.22

Therefore, the equation of sphere is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

General equation of sphere

We know that The equation of sphere having center at (a, b , c) and radius r is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

After expanding we get

$$x^2+y^2+z^2-2ax-2by-2cz+a^2+b^2+c^2=r^2$$

Or, $x^2+y^2+z^2+2(-a)x+2(-b)y+2(-c)z+\{a^2+b^2+c^2-r^2\}=0$

Or, $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Where, $u=(-a), v=(-b), w=(-c)$ and $d=\ a^2+b^2+c^2-r^2$

The centre of the sphere is $(-u\ ,-v,-w)$ and radius $r=\sqrt{u^2+v^2+w^2-d}$

Therefore, the general equation of sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

with centre at $(-u\ ,-v,-w)$, radius $r=\sqrt{u^2+v^2+w^2-d}$

Example: Find centre and radius of the sphere$x^2+y^2+z^2+4x+2y-6z-1=0$.

Sol: Equation of a sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

Now comparing $2u=4 \quad 2v=2 \quad 2w=-6$

$u=2 \quad v=1 \quad w=-3$ and $d=-1$

So, centre is $(-u,-v,-w)=(-2,-1,3)$

radius is, $r=\sqrt{u^2+v^2+w^2-d}$

$=\sqrt{4+1+9+1}$

$=\sqrt{15}$

Notes:

I. If $u^2+v^2+w^2=d$, then the radius of the sphere is equal to zero and the sphere is known as point sphere.
II. If $u^2+v^2+w^2<d$, then the sphere is known as Imaginary sphere.
III. If $u^2+v^2+w^2>d$, then the sphere is known as real sphere.
IV. Since there are four arbitrary constants namely u, v, w and d present in the equation of sphere, four conditions must be given in order to find equation of sphere .
V. Every second degree equation in x, y and z represents a sphere only when
   i. The co efficient of $x^2\ ,y^2 and\ z^2$ must be one or equal
   ii. The equation has no product term of x,y and z.
VI. If the sphere passes through origin , then its equation reduces to

$$x^2+y^2+z^2+2ux+2vy+2wz=0$$

(In that case there is no constant term, i.e $d=0$)

Equation of sphere passing through four given points

Let the sphere pass through the points $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ and $S(x_4, y_4, z_4)$.

Let equation of sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Since the sphere passes through the above four points, we get

Or, $x_1{}^2 + y_1{}^2 + z_1{}^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$

Or, $x_2{}^2 + y_2{}^2 + z_2{}^2 + 2ux_2 + 2vy_2 + 2wz_2 + d = 0$

Or, $x_3{}^2 + y_3{}^2 + z_3{}^2 + 2ux_3 + 2vy_3 + 2wz_3 + d = 0$

Or, $x_4{}^2 + y_4{}^2 + z_4{}^2 + 2ux_4 + 2vy_4 + 2wz_4 + d = 0$

Solving above four equations we get the value of $, v , w \ and \ d$.

Substituting the value of $u , v , w , and \ d$ in

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

We get the required equation of sphere.

EXERCISE

1. **02 Mark Questions**

i. Find the distance of the point P(1,2,3) from z axis.
ii. Find the direction cosines of the line joining the points (8 ,-1 ,5) and (2 , -4 ,3).
iii. Determine the direction cosines of the line equally inclined to both the axes.
iv. Find the number of lines making equal angles with coordinate axes.
v. If a line a line is perpendicular to z-axis and makes an angle measuring 60^0 with x –axis then find the angle it makes with y-axis.
vi. Find the projection of line segment joining (1,3,-1) and (3,2,4) on z axis.
vii. What is the image of the point (2,-4,7) with respect to xz plane.
viii. For what value of z, the distance between the points (-1,1,2) and (-1,-1,z) is 4.
ix. Find the centre of the sphere $x^2+y^2+(z+2)^2 =0$.
x. If the centre and radius of a sphere are (1,0,0) and 2 respectively, then find the equation of the sphere.
xi. If the segment of line joining the points (1,0,0) and (0,0,1) is a diameter of a sphere, then find equation of the sphere.

2. **05 Mark Questions**

i. Prove that angle between two main diagonals of cube is $\cos^{-1}\frac{1}{3}$
ii. Find the ratio in which the line through (1 ,-1 ,3) and (2 , -4 ,1) is divided by XY- plane.
iii. Find the ratio in which the line through (1 ,-1 ,3) and (2 , -4 ,1) is divided by YZ- plane.
iv. If P(x , y , 2) lies on the line through (1 ,-1 ,0) and (2 , 1 ,1). Find the values of x and y.
v. Find the ratio in which the line joining the points (2 , -3 , 1) and (3 , -4 , -5) is divided by the locus $2x - y + 3z - 4 = 0$.
vi. Find the foot of perpendicular drawn from the point (1 ,1 , 2) on the line joining (1 ,4 ,2) and (2 , 3 ,1).
vii. Find the value of k, if the distance between the points (-1 ,-1 ,k) and (1 ,-1 ,1) is 2.
viii. Find the value of 'a' such that two planes $2x + y + az - 2 = 0$ and $3x - y + 5z - 2 = 0$ are perpendicular to each other.
ix. Find angle between the planes $3x - y + 5z - 2 = 0$ and $3x - y + 5z - 2 = 0$
x. Find the equation of a plane passing through the points (1,2,3), (1,-2,-3) and perpendicular to the plane $3x - 3y + 5z - 2 = 0$.
xi. Find the equation of plane passing through intersection of planes $3x + y + z - 2 = 0$ and $x - 2y + 3z - 1 = 0$ and parallel to the plane $x - y + z - 6 = 0$.
xii. Find the equation of plane passing through intersection of planes $3x + 2y + z + 2 = 0$ and $x - 2y + 2z - 3 = 0$ and perpendicular to the plane $4x - y + 3z - 7 = 0$.
xiii. Find the equation of plane passing through the points (1,-1,-2) and perpendicular to the planes $4x - 2y + 3z - 1 = 0$ and $x + 2y + 3z - 2 = 0$
xiv. Find the equation of plane passing through the points (1,-2,3), (1,-1,-3) and (1,-3,0).
xv. Show that the points (1,2,3), (-1,1,0) (2,1,3) and (1,1,2) are coplanar.
xvi. Find the equation plane passing through the point (2, 3, -1) and parallel to the plane $x - y + z - 6 = 0$.
xvii. Find the equation of plane passing through the foot of the perpendicular drawn from points (1,2,3) on the co-ordinate planes .
xviii. Find the distance between parallel planes $2x - 3y + 6z + 1 = 0$ and $4x - 6y + 12z - 5 = 0$

xix. Find the equation of a sphere having centre at $(2,-1,\ 4)$ and the sphere touches the plane $2x-y-2z+6=0$

xx. Find the condition that the sphere $x^2+y^2+z^2-2x-2y-2z-6=0$ will touch the plane $x+y+z-a=0$

xxi. Find the equation of a sphere passing through the points $(0,0,0)$, $(0,1,-1)$, $(-1,\ 2,\ 0)$ and $(1,\ 2,\ 3)$

xxii. Find centre and radius of the sphere $x^2+y^2+z^2-x-y-z-6=0$ and $3x^2+3y^2+3z^2-4x+3y-z-6=0$

xxiii. Find the equation of a sphere having the two end points of a diameter as $(0,1,-1)$, $(-1,\ 2,\ 2)$.

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