DIGITAL SIGNAL PROCESSING

What is a Signal?

- Anything which carries information is a signal. e.g. human voice, chirping of birds, smoke signals, gestures (sign language), fragrances of the flowers.
- Modern high speed signals are: voltage changer in a telephone wire, the electromagnetic field emanating from a transmitting antenna,variation of light intensity in an optical fiber.
- Thus we see that there is an almost endless variety of signals and a large number of ways in which signals are carried from on place to another place.

Signals: The Mathematical Way

- A signal is a real (or complex) valued function of one or more real variable(s).When the function depends on a single variable, the signal is said to be one-dimensional and when the function depends on two or more variables, the signal is said to be multidimensional.
- Example of a one dimensional signal: A speech signal, daily maximum temperature, annual rainfall at a place An example of a two dimensional signal: An image is a two

dimensional signal, vertical and horizontal coordinates representing the two dimensions.
Four Dimensions: Our physical world is four dimensional(three spatial and one temporal).

Signal processing

- Processing means operating in some fashion on a signal to extract some useful information e.g. we use our ears as input device and then auditory pathways in the brain to extract the information.

  The signal is processed by a system.

- The signal processor may be an electronic system, a mechanical system or even it might be a computer program.

- The signal processing operations involved in many applications like communication systems, control systems, instrumentation, biomedical signal processing etc can be implemented in two different ways

Analog or continuous time method

Digital or discrete time method..

Analog signal processing

- Uses analog circuit elements such as resistors, capacitors, transistors, diodes etc

- Based on natural ability of the analog system to solve differential equations that describe a physical system
- The solutions are obtained in real time.

Digital signal processing

- The word digital in digital signal processing means that the processing is done either by a digital hardware or by a digital computer.
- Relies on numerical calculations.
- The method may or may not give results in real time.

Applications of Digital Signal Processing

- Speech Processing
- Image Processing
- Radar Signal Processing
- Digital Communications
- Optical Fiber Communications
- Telecommunication Networks
- Industrial Noise Control

The advantages of digital approach over analog approach

- Flexibility: Same hardware can be used to do various kind of signal processing operation, while in the case of analog signal processing one has to design a system for each kind of operation
- Repeatability: The same signal processing operation can be repeated again and again giving same results, while in analog systems there may be parameter variation due to change in temperature or supply voltage.
- Accuracy
- Easy Storage
- Mathematical Processing
- Cost
- Adaptability

Classification of signals

We use the term signal to mean a real or complex valued function of real variable(s) and denote the signal by $x(t)$ The variable t is called independent variable and the value x of t as dependent variable. When t takes a vales in a countable set the signal is called a discrete time signal.

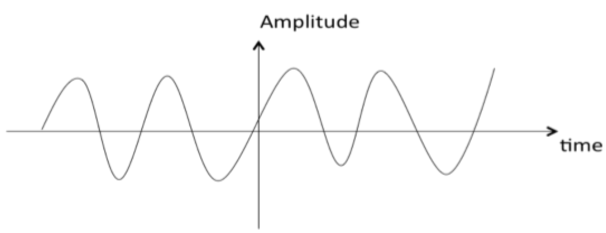
For convenience of presentation we use the notation $x[n]$ to denote discrete time signal.

When both the dependent and independent variables take values in countable sets (two sets can be quite different) the signal is called Digital Signal.

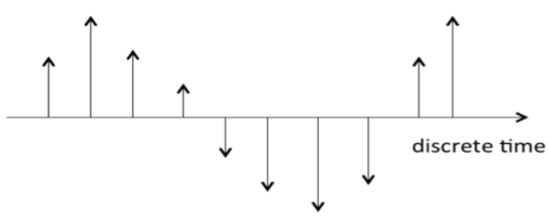
When both the dependent and independent variable take value in continous set interval, the signal is called an Analog Signal.

Continuous Time and Discrete Time Signals

- A signal is said to be continuous when it is defined for all instants of time.

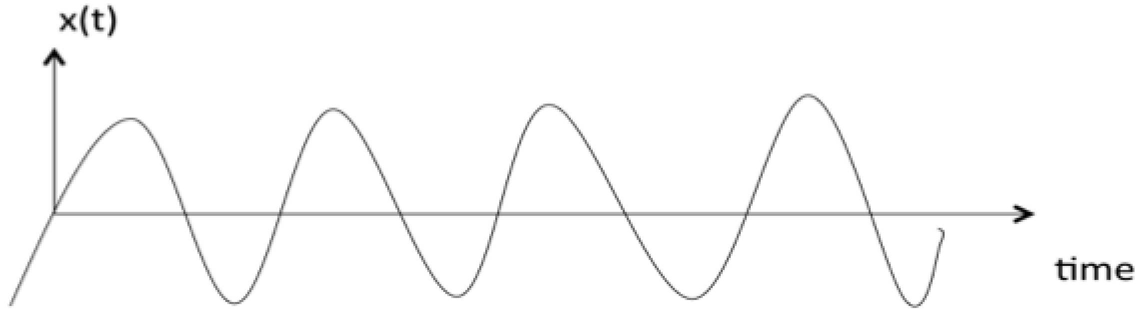


- A signal is said to be discrete when it is defined at only discrete instants of time.

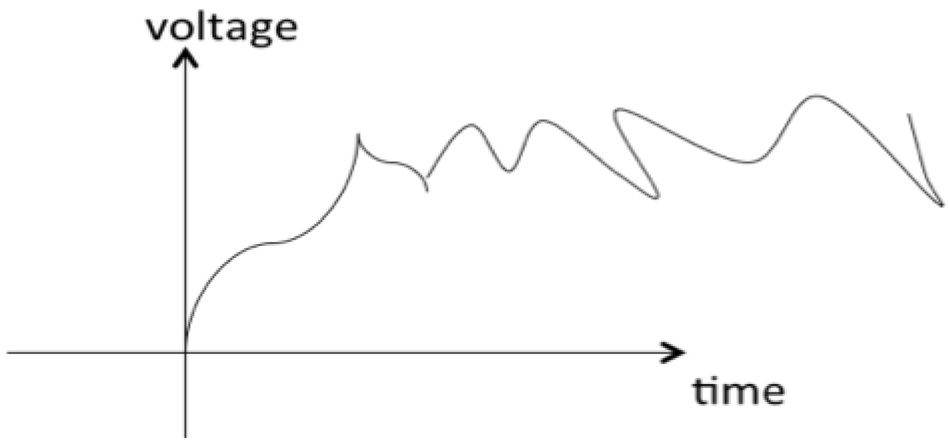


Deterministic and Non-deterministic Signals

A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time. Or, signals which can be defined exactly by a mathematical formula are known as deterministic signals.



A signal is said to be non-deterministic if there is uncertainty with respect to its value at some instant of time. Non-deterministic signals are random in nature hence they are called random signals. Random signals cannot be described by a mathematical equation. They are modelled in probabilistic terms.



Even and Odd Signals

A signal is said to be even when it satisfies the condition $x(t) = x(-t)$

Example 1:

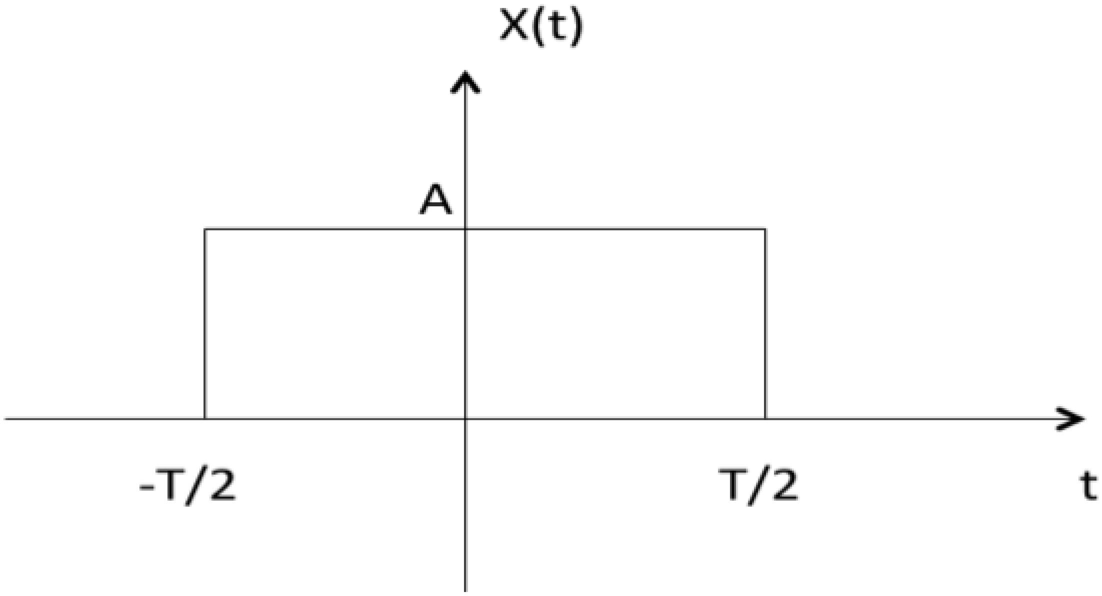
t^2, t^4… $\cos(t)$ etc.

Let $x(t) = t^2$

$x(-t) = (-t)^2 = t^2 = x(t)$

Thus t^2 is an even function

Example 2: As shown in the following diagram, rectangle function x(t) = x(-t) so it is also even function.



A signal is said to be odd when it satisfies the condition x(t) = -x(-t)

Example: t, t^3 ... And sin(t)

Let x(t) = sin t

x(-t) = sin(-t) = -sin t = -x(t)

Thus sin(t) is an odd function.

Any function ƒ(t) can be expressed as the sum of its even function $ƒ_e(t)$ and odd function $ƒ_o(t)$.

$ƒ(t) = ƒ_e(t) + ƒ_o(t)$

where

$ƒ_e(t) = ½ [ƒ(t) + ƒ(-t)]$

and $f_o(t) = ½ [f(t) - f(-t)]$

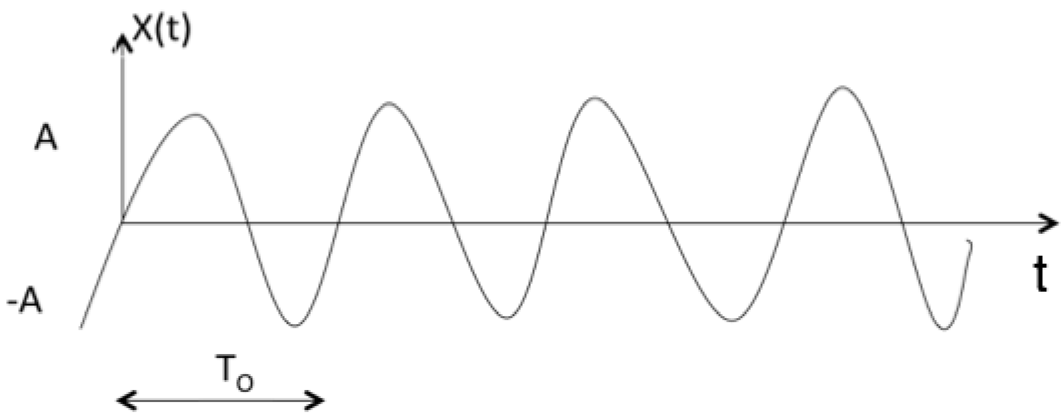
Periodic and Aperiodic Signals

A signal is said to be periodic if it satisfies the condition x(t) = x(t + T) or x(n) = x(n + N).

Where

T = fundamental time period,

1/T = f = fundamental frequency.



The above signal will repeat for every time interval T_0 hence it is periodic with period T_0.

Energy and Power Signals

A signal is said to be energy signal when it has finite energy.

$$\text{Energy, } E = \sum_{n=-\infty}^{+\infty} |x(n)|^2$$

A signal is said to be power signal when it has finite power.

Power, P= $\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2$

NOTE:A signal cannot be both, energy and power simultaneously. Also, a signal may be neither energy nor power signal.

Power of energy signal = 0

Energy of power signal = ∞

Energy/Power Signal Problems

Find the Energy and Power of the following signals and find whether the signals are power ,energy or neither energy nor power signals

1. **$x(n)=(1/3)^n u(n)$**

   Energy of the signal is given by

   E= $\sum_{n=-\infty}^{+\infty}|x(n)|^2$

   $=\sum_{n=-\infty}^{+\infty}[(1/3)^n]^2$

   $=\sum_{n=-\infty}^{\infty}(1/9)^n$

   $=\frac{1}{1-(\frac{1}{9})}$ $\quad\quad$ $1+a+a^2+a^3+........+\infty=\frac{1}{1-a}$

   $=\frac{9}{8}$

   Power of the signal is given by

$$P=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=0}^{N}\left(\frac{1}{9}\right)^{n}$$

$$=\lim_{n\to\infty}\frac{1}{2N+1}\frac{1-(\frac{1}{9})^{N+1}}{1-(\frac{1}{9})}$$

$$=0$$

So Energy is finite and Power is zero. Therefore the signal is an Energy signal.

2. **x(n)=e^{2n}u(n)**

$$E=\sum_{n=-\infty}^{+\infty}|x(n)|^2$$

$$=\sum_{n=0}^{\infty}e^{4n}$$

$$=\infty$$

$$P=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^{2}$$

$$=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}e^{4n}$$

$$=\lim_{N\to\infty}\frac{1}{2N+1}\frac{e^{4(N+1)-1}}{e^{4}-1}$$

$$=\infty$$

This signal is neither energy nor power signal.

<u>Periodic/ Aperiodic Signal Problem</u>

Determine whether or not each of the following signals is periodic . If periodic find its fundamental period

1. x(n)=$e^{j6\pi n}$=$e^{jw_0 n}$

So w_0=6π

Fundamental frequency is multiple of pi. So the signal is periodic.

Period of the signal is given by

N=2π $\frac{m}{w_0}$

= 2π $\frac{m}{6\pi}$

The minimum value of m for which N is an integer is 3

N= 2π $\frac{3}{6\pi}$

=1

Therefore the fundamental period is 1

2. x(n)=$e^{j3/5(n+1/2)}$

Here w_0=3/5,which is not a multiple of pi. So signal is aperiodic.

3. x(n)=cos (2π/3)n

Here w_0=2π/3.So periodic.

The fundamental period is

N=2π ($\frac{m}{2\pi/3}$)

=3m

For m=1, N=3

Therefore the fundamental period of the signal is 3.

4. $x(n)=\cos(\frac{\pi}{3}n) + \cos(\frac{3\pi}{4}n)$

The fundamental period of the signal $\cos(\frac{\pi}{3}n)$ **is**

$N_1=2\pi\left(\frac{m}{\pi/3}\right)$

$N_1=6m$

For m=1, $N_1=6$

The fundamental period of the signal $\cos(\frac{3\pi}{4}n)$ **is**

$N_2=2\pi\left(\frac{m}{3\pi/4}\right)$

$N_2=8m/3$

For m=3, $N_2=8$

Now $\frac{N1}{N2}=\frac{6}{8}=\frac{3}{4}$

So $N=4N_1=3N_2=24$

So N=24

Operations On signals

Signal processing is a group of basic operations applied to an input signal resulting in another signal as output. The mathematical transformation from one signal to another is represented as

$$y(n)=T[x(n)]$$

The basic set of operations are

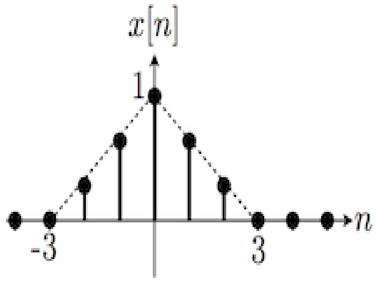
1. **Time Shifting**
2. **Time Reversal**
3. **Time Scaling**
4. **Amplitude Scaling**
5. **Signal Multiplier**
6. **Signal Addition**

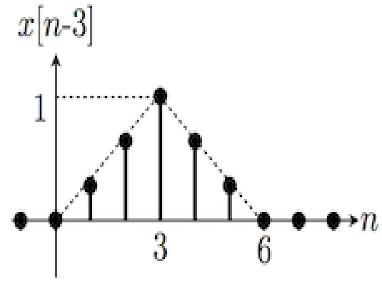
- **Time Shifting**

A signal x(n) may be shifted in time by replacing the independent variable n by n - k, where k is an integer.

$y(n)=x(n-k)$

- **If k is a positive integer, the time shift results in a delay of the signal by k units of time.**
- **If k is a negative integer, the time shift results in an advance of the signal by |k| units in time.**
  x(n-3) - Delay (Right Shift)
  x(n+3) – Advance (Left Shift)





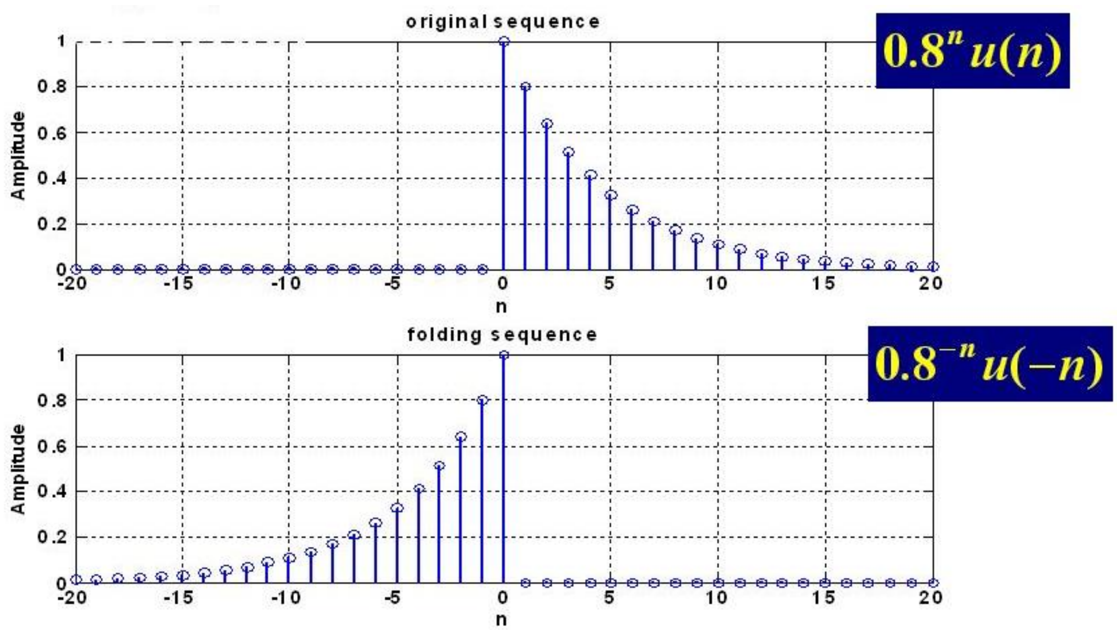
x(n-3) is obtained by shifting x(n) by 3 units towards right.

x(n+3) is obtained by shifting x(n) by 3 units towards left.

- **Time Reversal**

Another useful modification of the time base is to replace the independent variable n by -n. The result of this operation is a folding or a reflection of the signal about the time origin n = 0.

  - It is denoted as x(-n).



- **Time Scaling**

  A third modification of the independent variable involves replacing n by λn, where λ is an integer. We refer to this time-base modification as time scaling or downsampling.

  y(n)=x(λn)

Let x(n)= {0,0,0.25,0.75,1,0.75,0.25,0,0}

↑

When n=-3, x(2n)=x(-6)=0

When n=-2, x(2n)=x(-4)=0

When n=-1, x(2n)=x(-2)=0.25

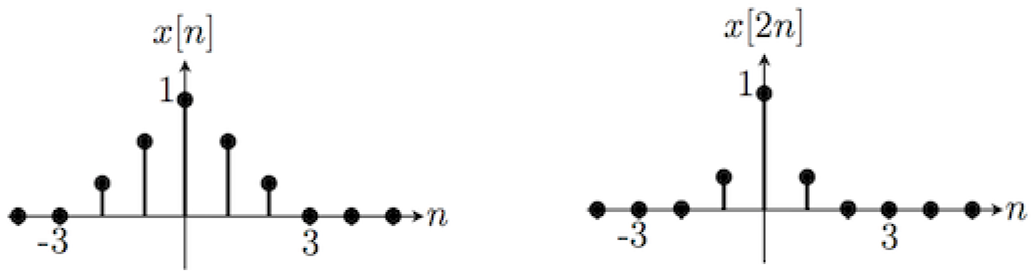
When n=-0, x(2n)=x(0)=1

When n=1, x(2n)=x(2)=0.25

When n=2, x(2n)=x(4)=0   and so on....

So x(2n)={0,0,0.25,1,0.25,0,0}

Graphically we can represent it as



- **Amplitude Scaling**
  - Amplitude modifications include addition, multiplication, and scaling of discrete-time signals.
  - Amplitude scaling of a signal by a constant A is accomplished by multiplying the value of every signal sample by A. Consequently, we obtain y(n) = A x(n)   — ∞< n < +∞

    e.g.

Let x(n)={ 1,2,3,4,5}

Then 2x(n) will be obtained simply by multiplying each sample of x(n) with 2

So 2x(n) ={2,4,6,8,10}

- **Signal Multiplier**

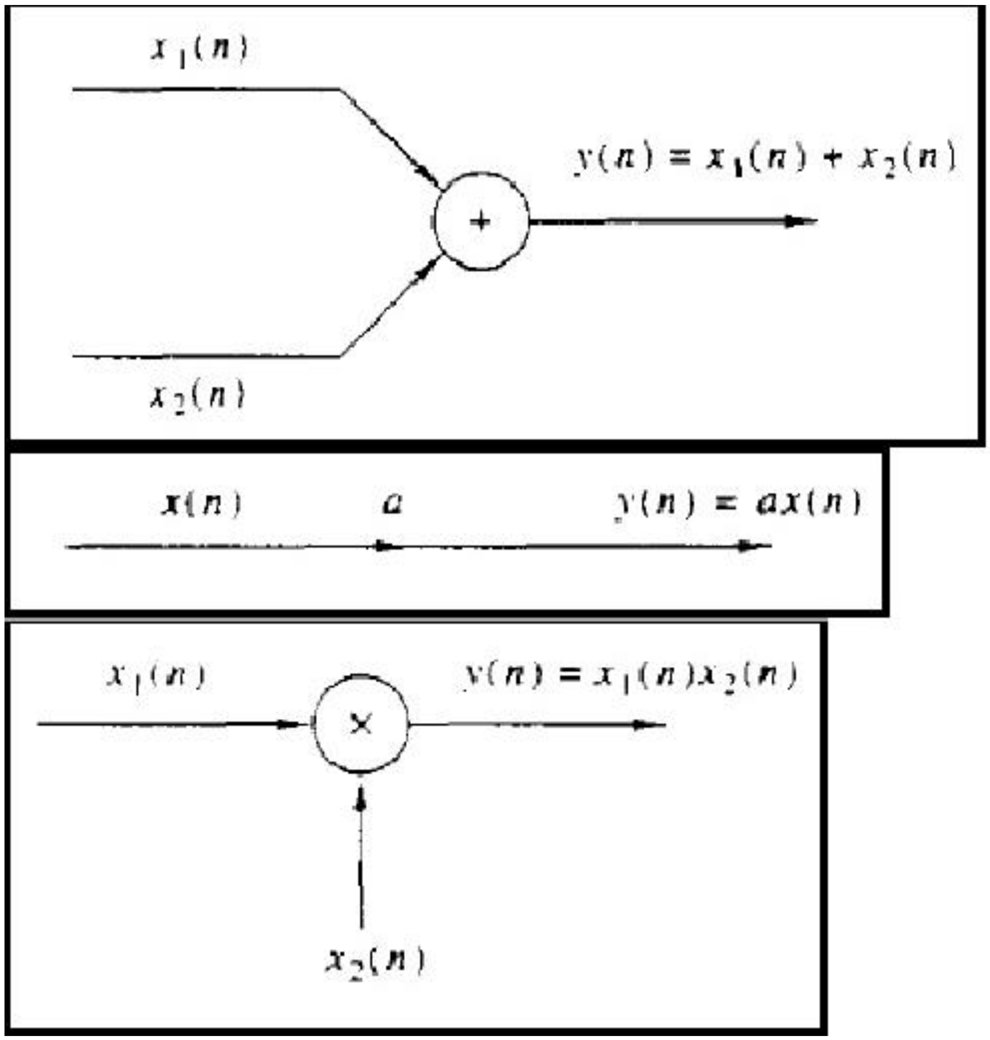
The product of two signals x1(n) and x2(n) is defined on a sample-to-sample basis as

$y(n) = x1(n) * x2(n) \qquad - \infty < n < +\infty$

- **Signal Addition**

The sum of two signals x1(n) and x2(n) is a signal y(n), whose value at any instant is equal to the sum of the values of these two signals at that instant, that is

$y(n) = x1(n) + x2(n). \qquad - \infty < n < +\infty$



DISCRETE-TIME SYSTEMS

- In many applications of digital signal processing we wish to design a device or an algorithm that performs some prescribed operation on a discrete-time signal.
- Such a device or algorithm is called a discrete-time system .
- More specifically, a discrete-time system is a device or algorithm that operates on a discrete-time signal, called the **input or excitation**, according to some well-defined

rule, to produce another discrete-time signal called the **output or response** of the system .

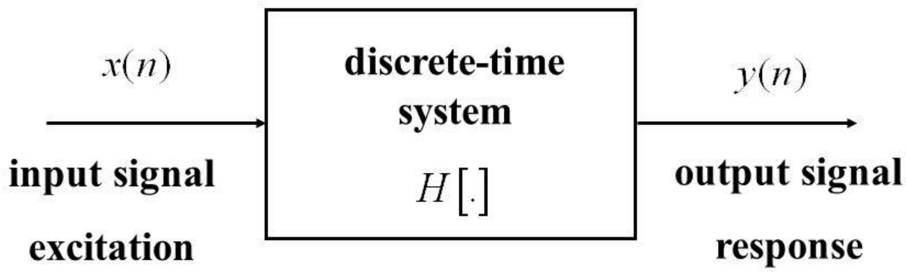
- In general, we view a system as an operation or a set of operations performed on the input signal x(n) to produce the output signal y(n). We say that the input signal x(n) is transformed by the system into a signal y(n), and express the general relationship between x(n) and y(n) as
    - $y(n) \equiv H[x(n)]$

Where the symbol H denotes the transformation (also called an operator), or processing performed by the system on x(n) to produce y(n).

The input output relation of a discrete time system can be shown by the below diagram.

Input-Output Model of Discrete-Time System

(input-output relationship description)



$$y(n) \equiv H[x(n)]$$

$$x(n) \xrightarrow{H} y(n)$$

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Question

Determine the response of the following sytems to the input signal

1. y(n) = x(n)
   In this case the output is exactly the same as the input signal. Such a system is known as the identity system
2. y(n) = x (n -1)
   This system simply delays the input by one sample.
3. y(n) = x(n+1)
   In this case the system “advances” the input one sample into the future

Block Diagram Representation of Discrete-Time Systems

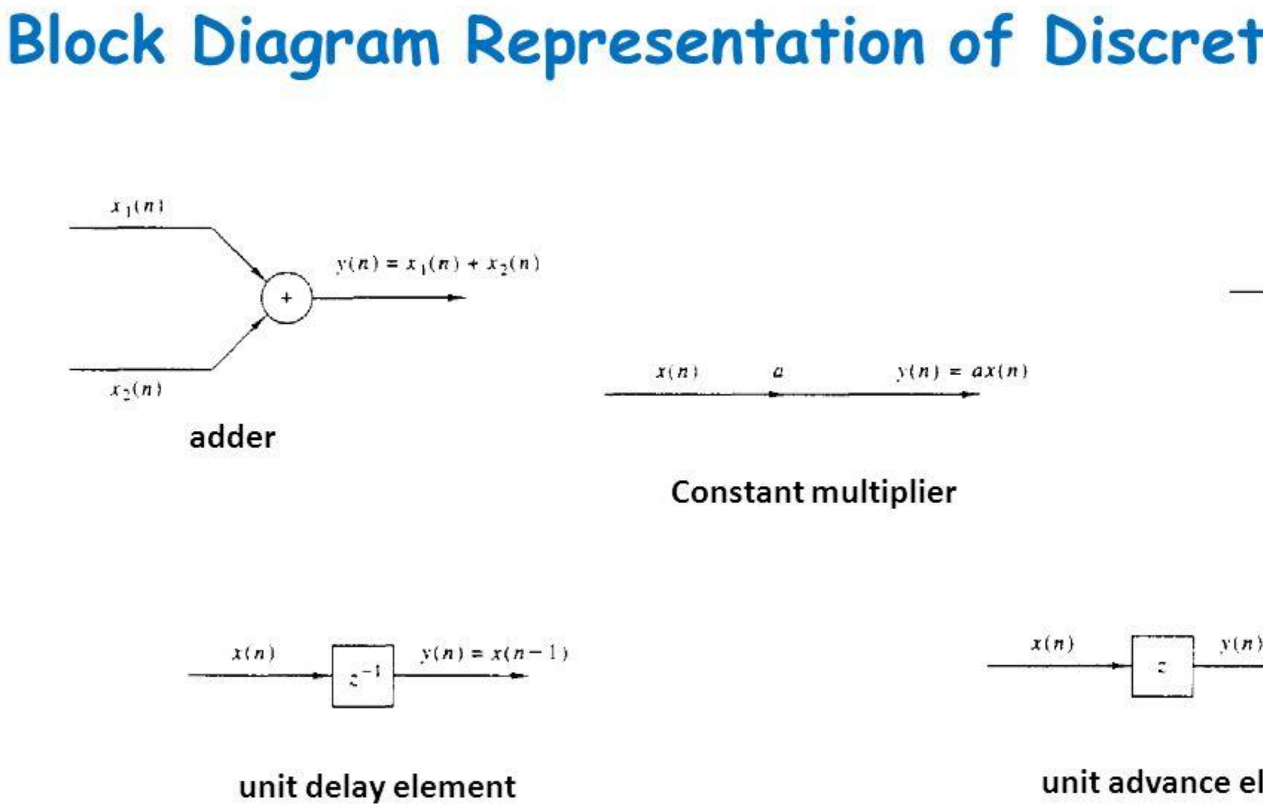
1. **An adder.** a system (adder) that performs the addition of two signal sequences to form another (the sum) sequence, which we denote as y(n).
   It is not necessary to store either one of the sequences in order to perform the addition. In other words, the addition operation is memoryless.

2. **A constant multiplier**. This operation represents applying a scale fact r on the input x(n). This operation is also memoryless.

3. **A signal multiplier**. The multiplication of two signal sequences to form another (the product) sequence y(n). The multiplication operation is also memoryless

4. **A unit delay element**. The unit delay is a special system that simply delays the signal passing through it by one sample. If the input signal is x(n), the output is x(n - 1).
   In fact, the sample x(n - 1) is stored in memory at time n - 1 and it is recalled from memory at time n to form y(n) = x(n - 1)

5. **A unit advance element**. In contrast to the unit delay, a unit advance moves the input x(n) ahead by one sample in time to yield x(n + 1).



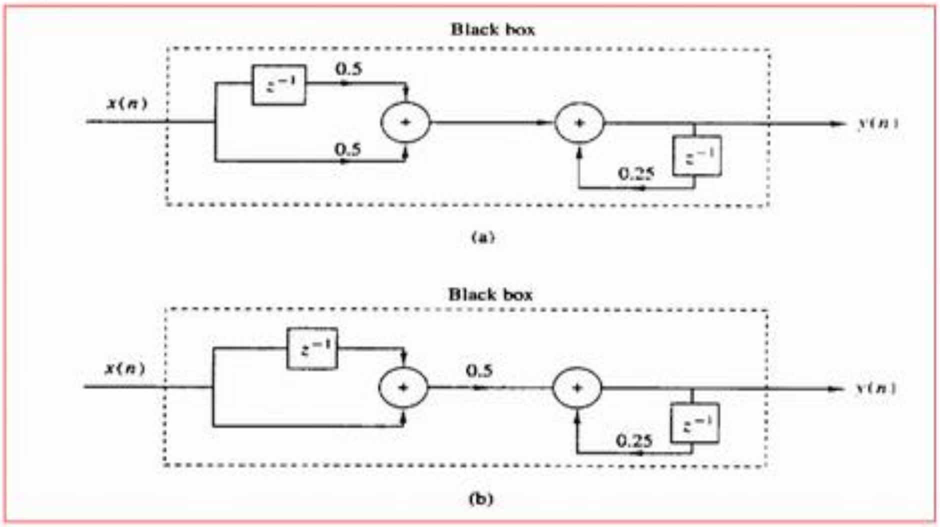
Question

Using basic building blocks introduced above, sketch the block diagram representation of the discrete-time system described by the input-output relation.

y(n)=1/4 y(n-1)+1/2 x(n)+1/2 x(n-1)

Where x(n) is the input and y(n) is the output of the system

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$$



Classification of Discrete-Time Systems

1. Static versus dynamic systems
2. Time-invariant versus time-variant systems
3. Linear versus nonlinear systems.
4. Causal versus noncausal systems
5. Stable versus unstable systems

<u>Static System Vs Dynamic System</u>

- A discrete-time system is called **static or memoryless** if its output at any instant n depends at most on the input sample at the same time, but not on past or future samples of the input.

- In any other case, the system is said to be **dynamic or to have memory**.

The systems described by the following input-output equations are both static or memoryless

$$y(n) = ax(n)$$
$$y(n) = nx(n) + bx^3(n)$$

On the other hand, the systems described by the following input-output relations are dynamic/having memory

$$y(n) = x(n) + 3\,x(n - 1)$$

$$y(n)=x(n)+x(n+2)$$

$y(n) = \sum_{k=0}^{n} x(n - k)$ Finite memory

$y(n) = \sum_{k=0}^{\infty} x(n - k)$ Infinite memory

So it can be said that static or memoryless systems are described in general by input-output equations of the form y(n) = T[x(n), n] and they do not include delay elements (memory).

<u>Time-invariant versus time-variant systems.</u>

- We can subdivide the general class of systems into the two broad categories, time-invariant systems and time-variant systems.
- A system is called **time-invariant** if its input-output characteristics do not change with time.
- To elaborate, suppose that we have a system T in a relaxed state which, when excited by an input signal x(n), produces an output signal y(n). Thus we write y(n) = T[x(n)]

Now suppose that the same input signal is delayed by k units of time to yield x(n - k), and again applied to the same system. If the characteristics of the system do not change with time, the output of the relaxed system will be y(n-k). That is, the output will be the same as the response to x(n), except that it will be delayed by the same k units in time that the input was delayed. This leads us to define a **time-invariant or shift-invariant** system as follows.

A relaxed system T is time invariant or shift invariant if and only if

$$x(n) \xrightarrow{T} y(n)$$

$$\Rightarrow x(n-k) \xrightarrow{T} y(n-k)$$

for every input signal $x(n)$ and every time shift k.

Linear versus nonlinear systems.

- The general class of systems can also be subdivided into linear systems and nonlinear systems.
- A **linear system** is one that satisfies the superposition principle.
- Simply stated, the principle of superposition requires that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of the responses (outputs) of the system to each of the individual input signals. Hence we have the following definition of linearity.
- A relaxed system is linear if and only if $\mathbf{T[a_1\, x_1\,(n) + a_2\, x_2\,(n)] = a_1\, T[x_1(n)] + a_2\, T[x_2(n)]}$ for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2
- The superposition principle embodied in the relation above can be separated into two parts.
- First, suppose that $a_2 = 0$. Then the above relation reduces to $T[a_1\, x_1\,(n)] = a_1\, T[x_1(n)] = a_1\, y_1(n)$ where $y_1(n) = T[x_1(n)]$
- The relation above demonstrates the multiplicative or scaling property of a linear system. That is, if the response of the system to the input $x_1\,(n)$ is $y_1(n)$, the response to $a_1\, x_1\,(n)$ is simply $a_1\, y_1(n)$.

- Thus any scaling of the input results in an identical scaling of the corresponding output.
- Second, suppose that $a_1 = a_2 = 1$. Then $T[a_1 x_1(n) + a_2 x_2(n)] = T[x_1(n)] + T[x_2(n)] = y_1(n)+y_2(n)$
- This relation demonstrates the **additivity** property of a linear system. The **additivity** and **multiplicative** properties constitute the superposition principle as it applies to linear systems.

Causal versus noncausal systems

- A system is said to be **causal** if the output of the system at any time n [i.e., y(n)] depends only on present and past inputs [i.e., x(n), x(n - 1), x(n- 2),...], but does not depend on future inputs [i.e., x(n + 1), x(n + 2),...]. In mathematical terms, the output of a causal system satisfies an equation of the form

$$\mathbf{y(n) = F[x(n), x(n-1), x(n-2),...]}$$

- If a system does not satisfy this definition, it is called **noncausal**. Such a system has an output that depends not only on present and past inputs but also on future inputs.

Stable versus unstable systems

- Stability is an important property that must be considered in any practical application of a system.
- Unstable systems usually exhibit erratic and extreme behavior and cause overflow in any practical implementation.
- An arbitrary relaxed system is said to be bounded input-bounded output (BIBO) stable if and only if every bounded input produces a bounded output.
- The conditions that the input sequence x(n) and the output sequence y(n) are bounded is translated mathematically to mean that there exist some finite numbers,
  say Mx and My such that $|x(n)|<M_x<\infty$ and $|y(n)|<M_y<\infty$ for all n.
- If for some bounded input sequence ,x(n), the output is unbounded (infinite), the system is classified as **unstable.**

PROBLEMS

Test the following systems for linearity

a) $y(n)=n\,x(n)$
b) $y(n)=x(n^2)$
c) $y(n)=x^2(n)$
d) $y(n)=e^{x(n)}$

Procedure

1. Let $x_1(n)$ and $x_2(n)$ be two inputs to system H and $y_1(n)$ and $y_2(n)$ be corresponding responses
2. Consider a signal $x_3(n)= a_1\ x_1(n)+a_2\ x_2(n)$ which is a weighted sum of $x_1(n)$ and $x_2(n)$.
3. Let $y_3(n)$be the response for $x_3(n)$.
4. Check whether $y_3(n)= a_1\ y_1(n)+a_2\ y_2(n)$.If they are equal then the system is linear, otherwise it is nonlinear.

Solution

a. Consider 2 signals $x_1(n)$ and $x_2(n)$

Let $y_1(n)$ and $y_2(n)$ be the response of the system H for inputs $x_1(n)$ and $x_2(n)$ respectively

$y_1(n)=H\{ x_1(n) \}= n\ x_1(n)$

$y_2(n)=H\{ x_2(n) \}= n\ x_2(n)$

So $a_1\ y_1(n)+a_2\ y_2(n)= a_1\ n\ x_1(n)+a_2\ n\ x_2(n)$

Now consider a linear combination of inputs $x_3(n)= a_1\ x_1(n)+a_2\ x_2(n)$.

Let $y_3(n)$ be the response for this linear combination of inputs

$y_3(n)=H\{ a_1\ x_1(n)+a_2\ x_2(n)\}=n[a_1\ x_1(n)+a_2\ x_2(n)]= a_1\ n\ x_1(n)+a_2\ n\ x_2(n)$

Since $y_3(n)= a_1\ y_1(n)+a_2\ y_2(n)$,the system is **linear**.

b. Consider 2 signals $x_1(n)$ and $x_2(n)$

Let $y_1(n)$ and $y_2(n)$ be the response of the system H for inputs $x_1(n)$ and $x_2(n)$ respectively

$y_1(n)=H\{ x_1(n) \}= x_1(n^2)$

$y_2(n)=H\{ x_2(n) \}= x_2(n^2)$

So $a_1 y_1(n)+a_2 y_2(n)= a_1 x_1(n^2)+a_2 x_2(n^2)$

Now consider a linear combination of inputs $x_3(n)= a_1 x_1(n)+a_2 x_2(n)$.

Let $y_3(n)$ be the response for this linear combination of inputs.

$y_3(n)=H\{ x_3(n)\}=H\{ a_1 x_1(n)+a_2 x_2(n)\}= a_1 x_1(n^2)+a_2 x_2(n^2)$

Since $y_3(n)= a_1 y_1(n)+a_2 y_2(n)$,the system is **linear**

c. Consider 2 signals $x_1(n)$ and $x_2(n)$

Let $y_1(n)$ and $y_2(n)$ be the response of the system H for inputs $x_1(n)$ and $x_2(n)$ respectively

$y_1(n)=H\{ x_1(n) \}= x_1^2(n)$

$y_2(n)=H\{ x_2(n) \}= x_2^2(n)$

So $a_1 y_1(n)+a_2 y_2(n)= a_1 x_1^2(n)+a_2 x_2^2(n)$

Now consider a linear combination of inputs $x_3(n)= a_1 x_1(n)+a_2 x_2(n)$.

Let $y_3(n)$ be the response for this linear combination of inputs.

$y_3(n)= H\{ x_3(n)\}=H\{ a_1 x_1(n)+a_2 x_2(n)\}= x_3^2(n)= [a_1 x_1(n)+a_2 x_2(n)]^2=a_1^2 x_1^2(n)+ a_2^2 x_2^2(n)+2 a_1 x_1(n) a_2 x_2(n)$

Since $y_3(n)\neq a_1 y_1(n)+a_2 y_2(n)$,the system is **non-linear**.

d. Consider 2 signals $x_1(n)$ and $x_2(n)$

Let $y_1(n)$ and $y_2(n)$ be the response of the system H for inputs $x_1(n)$ and $x_2(n)$ respectively

$y_1(n)=H\{ x_1(n) \}= e^{x1(n)}$

$y_2(n)=H\{ x_2(n) \}= e^{x2(n)}$

So $a_1 y_1(n)+a_2 y_2(n)= a_1 e^{x1(n)}+a_2 e^{x2(n)}$

Now consider a linear combination of inputs $a_1 x_1(n)+a_2 x_2(n)$.

Let $y_3(n)$ be the response for this linear combination of inputs

$y_3(n)= H\{ x_3(n)\}=H\{ a_1 x_1(n)+a_2 x_2(n)\}= e^{x3(n)}=e^{[a1\ x1(n)+a2\ x2(n)]}= e^{a1x1(n)}\ e^{a2x2(n)}$

Since $y_3(n)\neq a_1 y_1(n)+a_2 y_2(n)$,the system is **non-linear**

Test the following system for linearity

a) $y(n)=2\ x(n)+\frac{1}{x(n-1)}$

b) $y(n)=x(n)-b\ x(n-1)$

Solution

a. Consider 2 signals $x_1(n)$ and $x_2(n)$

Let $y_1(n)$ and $y_2(n)$ be the response of the system H for inputs $x_1(n)$ and $x_2(n)$ respectively

$y_1(n)=H\{ x_1(n) \}= 2x_1(n)+ \frac{1}{x1(n-1)}$

$y_2(n)=H\{ x_2(n) \}= 2x_2(n)+ \frac{1}{x2(n-1)}$

So $a_1\, y_1(n)+a_2\, y_2(n)= a_1\, [2x_1(n)+ \frac{1}{x1(n-1)}]+a_2\, [2x_2(n)+ \frac{1}{x2(n-1)}]$

Now consider a linear combination of inputs $a_1\, x_1(n)+a_2\, x_2(n)$.

Let $y_3(n)$ be the response for this linear combination of inputs

$y_3(n)=H\{ a_1\, x_1(n)+a_2\, x_2(n)\}= 2[a_1\, x_1(n)+a_2\, x_2(n)]+ \frac{1}{a1\, x1(n-1)+a2\, x2(n-1)}$

Since $y_3(n)\neq a_1\, y_1(n)+a_2\, y_2(n)$,the system is **non-linear**

b. Consider 2 signals $x_1(n)$ and $x_2(n)$

Let $y_1(n)$ and $y_2(n)$ be the response of the system H for inputs $x_1(n)$ and $x_2(n)$ respectively

$y_1(n)=H\{ x_1(n) \}= x_1(n)-bx_1(n-1)$

$y_2(n)=H\{ x_2(n) \}= x_2(n)-bx_2(n-1)$

So $a_1\, y_1(n)+a_2\, y_2(n)= a_1\, x_1(n)- a_1bx_1(n-1)+ a_2\, x_2(n)- a_2bx_2(n-1)$

Now consider a linear combination of inputs $a_1\, x_1(n)+a_2\, x_2(n)$.

Let $y_3(n)$ be the response for this linear combination of inputs

$y_3(n)$=H{ a_1 $x_1(n)$+a_2 $x_2(n)$}= a_1 $x_1(n)$+a_2 $x_2(n)$-b [a_1 $x_1(n-1)$+a_2 $x_2(n-1)$]

=a_1 $x_1(n)$- $a_1bx_1(n-1)$+ a_2 $x_2(n)$- $a_2bx_2(n-1)$

Since $y_3(n)= a_1 y_1(n)+a_2 y_2(n)$,the system is **linear**

Response of LTI Discrete Time System in Time Domain

The general equation governing an LTI discrete time system is

$$y(n)=-\sum_{m=1}^{N} a_m y(n-m)+\sum_{m=0}^{M} b_m x(n-m)$$

$$\sum_{m=0}^{N} a_m y(n-m)=\sum_{m=0}^{M} b_m x(n-m) \text{ with } a_0=1$$

The solution of the difference equation is the response y(n) of LTI system, which consists of two parts. In mathematics, the two parts of the solution y(n) are homogeneous solution $y_h(n)$ and particular solution $y_p(n)$

Response, $y(n)= y_h(n)+ y_p(n)$

The homogeneous solution is the response of the system when there is no input.

The particular solution is the solution of difference equation for specific input signal x(n) for n≥0

In signals and systems, the two parts of the solution y(n) are called zero-input response $y_n(n)$ and zero-state response $y_{zs}(n)$

zero-input response

- The zero input response is mainly due to initial conditions in the system. Hence zero-input response is also called free response or natural response.
- The zero input response is given by homogeneous solution with constants evaluated using initial conditions.

zero-state response

- The zero-state response is the response of the system due to input signal and with zero initial condition. Hence the zero state response is called forced response. The zero state response or forced response is given by the sum of homogeneous solution and particular solution with zero

Question

Determine the response of first order discrete time system governed by the difference equation

$$y(n) = -0.5y(n-1) + x(n)$$

When the input is unit step, and with initial condition

a) **y(-1)=0**
b) **y(-1)=1/3**

Solution

$y(n)+0.5y(n-1)= x(n)$ ………………………………..(1)

Homogeneous Solution

The homogeneous equation is the solution of equation 1 when $x(n)=0$

$y(n)+0.5y(n-1)= 0$

Putting $y(n)=\lambda^n$ in the above equation

$\lambda^n+0.5\lambda^{n-1}=0$

$\lambda^{n-1}(\lambda+0.5)=0$

$\lambda+0.5=0$

$\Rightarrow \lambda=-0.5$

The homogeneous solution $y_h(n)$ is given by

$y_h(n)=C\lambda^n=C(-0.5)^n$ for $n\geq 0$ ……………………………….(2)

Particular Solution

Given that the input is unit step and so the particular solution will be in the form,

$y(n)=K\,u(n)$

Putting this in equation 1 we get

$K\,u(n)+0.5\,K\,u(n-1)=u(n)$ ………………………………….(3)

In order to determine the value of K, we have to evaluate for n=1 in equation 3

$K\,u(1)+0.5\,K\,u(0)=u(1)$

$K+0.5K=1$ [As $u(1)=1, u(0)=1$]

$1.5K=1$

$K=1/1.5=2/3$

The particular solution $y_p(n)$ is given by

$y_p(n)=K\,u(n)=2/3\,u(n)$ for all n

Total Response

The total response y(n) of the system is given by sum of homogeneous and particular solution

Response $y(n)= y_h(n)+ y_p(n)$

$= C(-0.5)^n + 2/3\ u(n)$

$= C(-0.5)^n + 2/3$ for $n \geq 0$ ……………………………(4)

At n=0, equation 1 becomes

$y(n)+0.5y(n-1)=x(n)$

$y(0)+0.5\ y(-1)=1$

5)

$y(0)=1-0.5y(-1)$

At n=0, equation 4 becomes

$y(0)=C+ 2/3$ ………………………………………….(6)

From equation (5) and (6) we get

$C+ 2/3 = 1-0.5y(-1)$

$C=1-0.5y(-1)-2/3$

$C=1/3-0.5\ y(-1)$

Putting the value of C in equation 4 we get

$y(n)=(1/3-0.5\ y(-1))\ (-0.5)^n +2/3$

a) When $y(-1)=0$

$y(n)=1/3\ (-0.5)^n + 2/3$ for $n \geq 0$

b) When $y(-1)=1/3$

$y(n)=[1/3 -0.5 \times 1/3]\ (-0.5)^n + 2/3$

$y(n)= 0.5/3\ (-0.5)^n + 2/3$ for $n \geq 0$

Determine the response y(n),n≥0 of the system described by the second order difference equation
$y(n)-2y(n-1)-3y(n-2)=x(n)+4x(n-1)$ ....(1)
when the input signal is $x(n)=2^n u(n)$ and with initial conditions $y(-2)=0, y(-1)=5$

Homogeneous Solution:

It is the solution when $x(n)=0$

$y(n)-2y(n-1)-3y(n-2)=0$ ....(2)

Putting $y(n)=\lambda^n$ in equation 2,we get

$\lambda^n-2\lambda^{n-1}-3\lambda^{n-2}=0$

$\lambda^{n-2}(\lambda^2-2\lambda-3)=0$

The characteristics equation is

$\lambda^2-2\lambda-3=0$

⇨ $(\lambda-3)(\lambda+1)=0$

The roots are $\lambda=3,-1$

The homogeneous solution, $y_h(n)=C_1\lambda_1^{\,n}+C_2\lambda_2^{\,n}=C_1(3)^n+C_2(-1)^n$

Particular Solution:

Let $y(n)=K\,2^n u(n)$

Putting $y(n)=K\,2^n u(n)$ in equation 1,we get

$K\ 2^n\ u(n)-\ 2K\ 2^{n-1}\ u(n-1)-3\ K\ 2^{n-2}\ u(n-2)=\ 2^n\ u(n)+4\ 2^{n-1}\ u(n-1)$ ....(3)

In order to find the value of K, we put n=2 in equation 3

$K\ 2^2\ u(2)-\ 2K\ 2^{2-1}\ u(2-1)-3\ K\ 2^{2-2}\ u(2-2)=\ 2^2\ u(2)+4\ 2^{2-1}\ u(2-1)$

- ⇨ 4K-4K-3K=4+4 x 2
- ⇨ -3K=12
- ⇨ K=-12/3=-4

So the particular solution $y_p(n)=K\ 2^n\ u(n)=(-4)\ 2^n\ u(n)$

Total Solution:

$y(n)=y_h(n)+y_p(n)$

$y(n)=C_1\ (3)^n+C_2\ (-1)^n+(-4)\ 2^n\ u(n)$ for $n\geq 0$................................(4)

When n=0, equation 1 becomes

y(0)-2y(0-1)-3y(0-2)=x(0)+4 x(0-1)

y(0)-2y(-1)-3y(-2)=x(0)+4 x(-1)..(5)

Given that y(-1)=5,y(-2)=0

$x(n)=2^n\ u(n)$

$n=0,x(0)=2^0\ u(0)=1$

$n=-1,x(-1)=2^{-1}\ u(-1)=0$

Putting the above conditions in equation 5 , we get

y(0)-2y(-1)-3y(-2)=x(0)+4 x(-1)

y(0)-2 x 5-3 x 0=1+4 x 0

$y(0)-10=1$

$y(0)=11$

When n=1, equation 1 becomes

$y(1)-2y(1-1)-3y(1-2)=x(1)+4\ x(1-1)$

$y(1)-2y(0)-3y(-1)=x(1)+4\ x(0)$ ....(6)

We know that $y(0)=11$, $y(-1)=5$

$x(n)=2^n\ u(n)$

$n=0, x(0)=2^0\ u(0)=1$

$n=1, x(1)=2^1\ u(1)=2$

Putting the above conditions in equation 6 , we get

$y(1)-2y(0)-3y(-1)=x(1)+4\ x(0)$

$y(1)-2 \times 11-3 \times 5=2+4 \times 1$

$y(1)-22-15=2+4$

$y(1)=43$

When n=0 ,

Equation 4 becomes

$y(0)= C_1 (3)^0 + C_2 (-1)^0 +(-4) 2^0 u(0) =C_1+C_2-4$ ....(7)

$C_1+C_2-4=11$

$C_1+C_2=15$ ....(8)

When n=1 ,

Equation 4 becomes

$y(1)= C_1 (3)^1 + C_2 (-1)^1 +(-4)\, 2^1 u(1) \quad =3C_1-C_2-8$

$3C_1-C_2-8=43$

$3C_1-C_2=51$ ....(9)

Adding equation 8 and 9 we get

$4C_1=66$

$C_1=66/4=33/2$

$C_2==3/2$

The total solution is

$$\mathbf{y(n)= 33/2\ (3)^n -3/2\ (-1)^n -4\ 2^n u(n) \quad \text{for } n\geq 0}$$

Response of LTI Systems to Arbitrary Inputs: The Convolution Sum

The formula in that gives the response y(n) of the LTI system as a function of the input signal x(n) and the unit sample (impulse) response h(n) is called a convolution sum.

$$y(n)=\sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

We say that the input x(n) is convolved with the impulse response h(n) to yield the output y(n).

If x(n) has N_1 samples and h(n) has N_2 samples, then the output sequence y(n) will have "N_1+N_2-1 " samples

The convolution relation can be symbolically expressed as

$$y(n)=x(n)*h(n)= h(n)*x(n)$$

Procedure for evaluating linear convolution

The process of computing the convolution between x(k) and h(k) involves the following four steps.

1. **Folding** Fold h(k) about k = 0 to obtain h(-k).
2. **Shifting** Shift h(-k) by n_0 to the right (left) if n_0 is positive (negative), to obtain $h(n_0 - k)$.
3. **Multiplication** Multiply x(k) by $h(n_0 - k)$ to obtain the product sequence $vn_0 (k) = x(k)h(n_0 - k)$.
4. **Summation** Sum all the values of the product sequence $vn_0 (k)$ to obtain the value of the output at time $n = n_0$.

The above procedure results in the response of the system at a single time instant, say $n = n_0$.

In general, we are interested in evaluating the response of the system over all time instants $-\infty < n < \infty$. Consequently, steps 2 through 4 in the summary must be repeated, for all possible time shifts $-\infty < n < \infty$.

Properties of Linear Convolution

The discrete convolution will satisfy the following properties

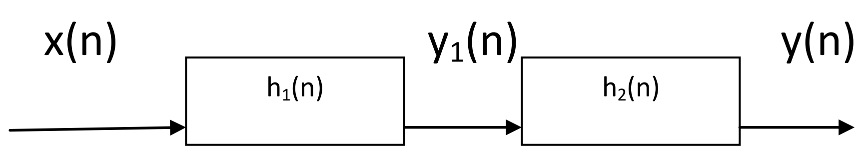
1. Commutative Property : **$x_1(n) * x_2(n) = x_2(n) * x_1(n)$**
2. Associative Property : **$[x_1(n) * x_2(n)] * x_3(n) = x_1(n) * [x_2(n) * x_3(n)]$**
3. Distributive Property : **$x_1(n) * [x_2(n) + x_3(n)] = [x_1(n) * [x_2(n)] + [x_1(n) * x_3(n)]$**

Interconnection of Discrete Time Systems

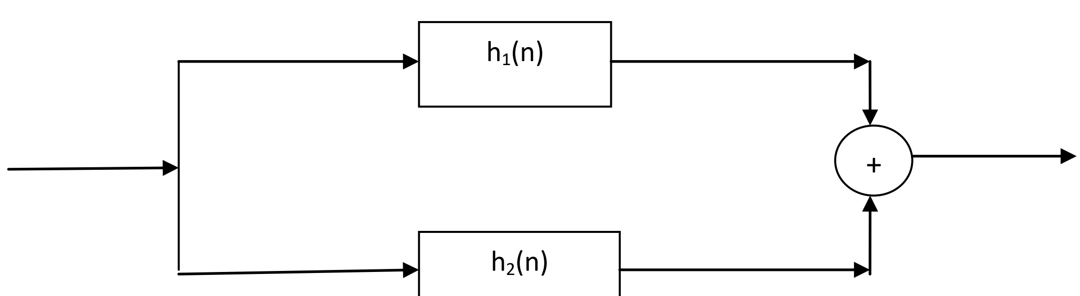
Smaller discrete time systems may be interconnected to form larger systems. Two possible basic ways of interconnection are Cascade connection and Parallel Connection.

The cascade and parallel connections of two discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ are given below.

Cascade Connection



Parallel Connection



x(n)
y(n)

Two cascade connected discrete time system with impulse response $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time sys… mpulse response is given by convolution of ind… responses.

$h_1(n) * h_2(n)$

x(n) → y(n) →

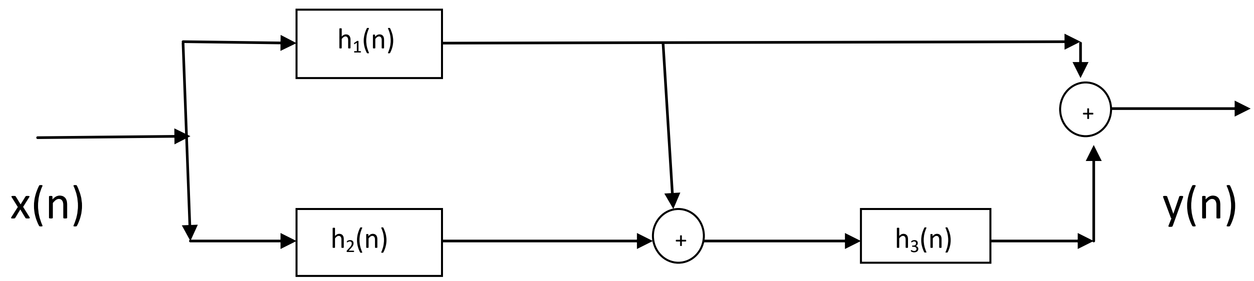
Two parallel connected discrete time system with impulse response $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time syst… ulse response is given by sum of individual impu…

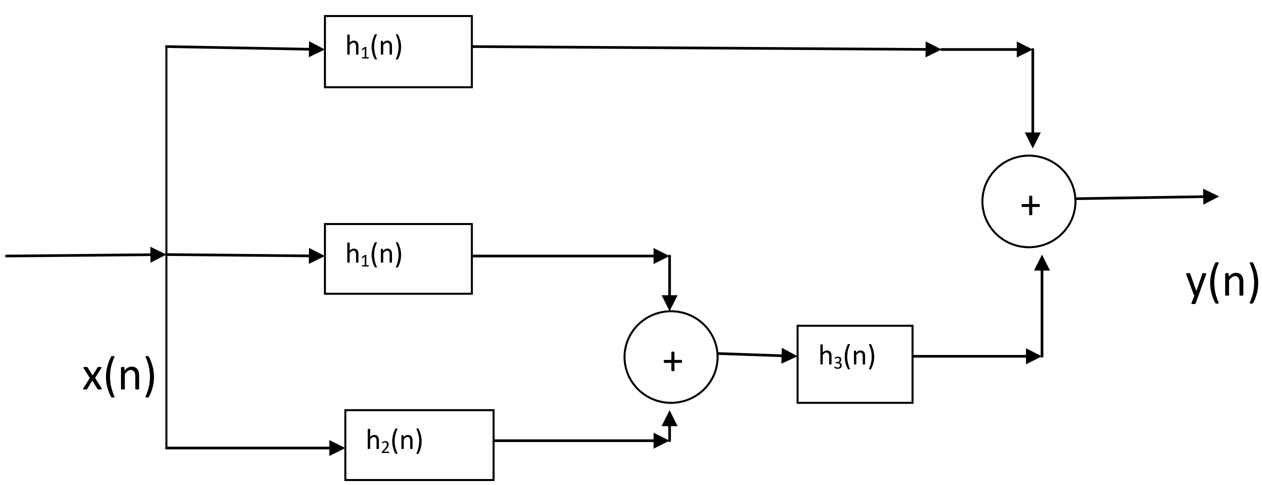
$h_1(n) + h_2(n)$

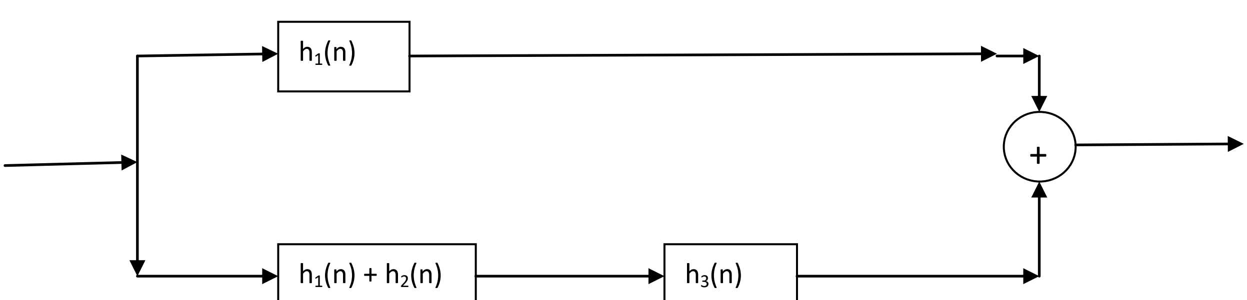
x(n) → y(n) →

Question

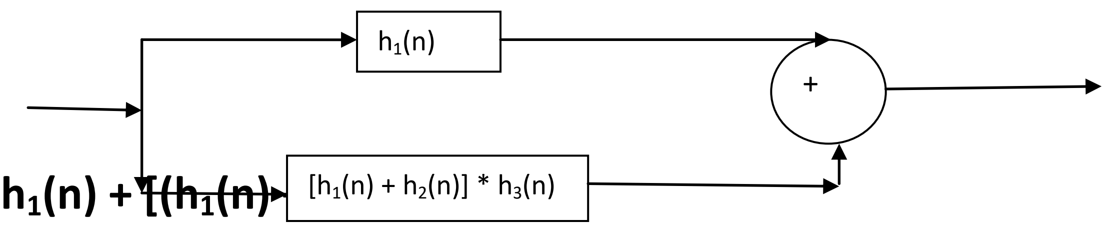
Simplify the overall impulse response of the interconnected discrete time system shown below







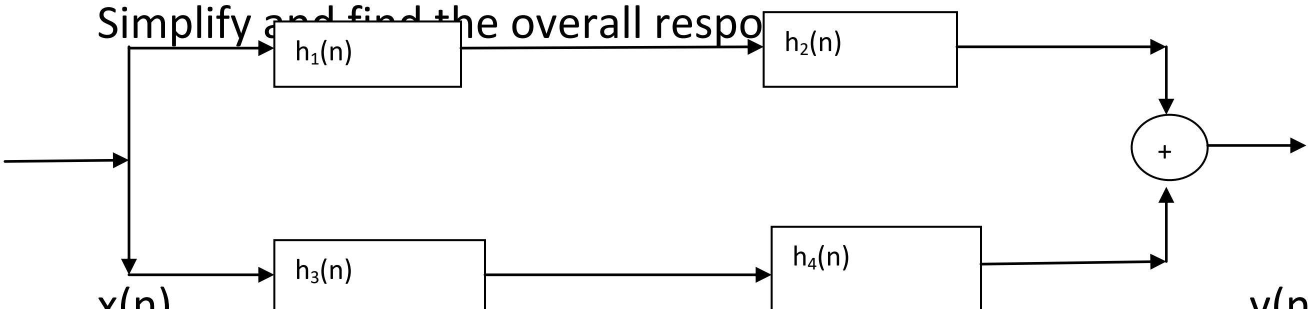
x(n) y(n)



h(n)=

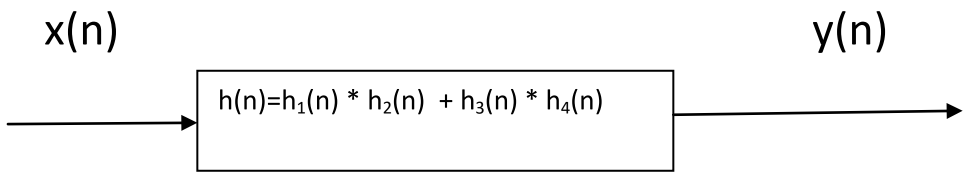
$h_1(n)$ + [($h_1(n)$

Simplify and find the overall respo



x(n) y(n)

$$h(n)=h_1(n) * h_2(n) + h_3(n) * h_4(n)$$



Correlation

Correlation is a measure of similarity between two signals. The general formula for correlation is

$$\int x_1(t)\, x_2(t-\tau)dt$$

There are two types of correlation:

- Auto correlation
- Cross correlation

Auto Correlation Function

It is defined as correlation of a signal with itself. Auto correlation function is a measure of similarity between a signal & its time delayed version. It is represented with R(ττ).

Cross Correlation Function

Cross correlation is the measure of similarity between two different signals.

The impulse response of an LTI system is h(n)={ 1,2 ,1,-1}.Find the response of the system for the input x(n)={ 1,2 ,3,1}

The response y(n) of the system is given by convolution of x(n) and h(n)

y(n)=x(n) * h(n) …………………………(1)

By convolution theorem of Fourier transform, we know that

F {x(n) * h(n)}= $X(e^{jw})$. $H(e^{jw})$ ……………………….(2)

From equation (1) and (2) we can write

F{y(n)}= $X(e^{jw})$. $H(e^{jw})$

Let F{y(n)}=$Y(e^{jw})$

$Y(e^{jw})= X(e^{jw}) . H(e^{jw})$

$y(n)=F^{-1}\{ X(e^{jw}) . H(e^{jw}) \}$

x(n)={ 1,2 ,3,1}

$X(e^{jw})= \sum_{n=-\infty}^{\infty} x(n)e^{-jwn}$

$X(e^{jw})= \sum_{n=0}^{3} x(n)e^{-jwn}$

$X(e^{jw})= x(0)e^{-jw0}+ x(1)e^{-jw1}+ x(2)e^{-jw2}+ x(3)e^{-jw3}$

$X(e^{jw})=1+2e^{-jw}+3e^{-j2w}+e^{-j3w}$

h(n)={ 1,2 ,1,-1}

$H(e^{jw})= \sum_{n=-\infty}^{\infty} h(n)e^{-jwn}$

$H(e^{jw})= \sum_{n=0}^{3} h(n)e^{-jwn}$

$H(e^{jw}) = h(0)e^{-jw0} + h(1)e^{-jw1} + h(2)e^{-jw2} + h(3)e^{-jw3}$

$H(e^{jw}) = 1+2e^{-jw}+e^{-j2w}-e^{-j3w}$

$X(e^{jw}) \,.\, H(e^{jw}) = (1+2e^{-jw}+3e^{-j2w}+e^{-j3w}) \,.\, (1+2e^{-jw}+e^{-j2w}-e^{-j3w})$

$=1+2e^{-jw}+e^{-j2w}-e^{-j3w}+ 2e^{-jw}(1+2e^{-jw}+e^{-j2w}-e^{-j3w})+ 3e^{-j2w}(1+2e^{-jw}+e^{-j2w}-e^{-j3w})+ e^{-j3w}(1+2e^{-jw}+e^{-j2w}-e^{-j3w})$

$Y(e^{jw})=1+4e^{-jw}+8e^{-j2w}+8e^{-j3w}+3e^{-j4w}-2e^{-j5w}-e^{-j6w}$ ....(3)

By definition of Fourier transform we get,

$Y(e^{jw}) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-jwn}$

$=1+4e^{-jw}+8e^{-j2w}+8e^{-j3w}+3e^{-j4w}-2e^{-j5w}-e^{-j6w}$

$=y(0)e^{0}+y(1)e^{-jw}+y(2)e^{-j2w}+y(3)e^{-j3w}+y(4)e^{-j4w}-y(5)e^{-j5w}-y(6)e^{-j6w}$...........(4)

Comparing equation (3) & (4) we get

$y(n)=\{1,4,8,8,3,-2,-1\}$

Discrete Fourier Transform(DFT) & Fast Fourier Transform(FFT)

- Discrete time Fourier transform(DTFT) is used to represent a discrete time signal in frequency domain and to perform frequency analysis of DT signals.
- Drawbacks:
  - Its frequency domain representation is a continuous function of ω
  - Can not be processed by digital system.

Discrete Fourier Transform(DFT)

- It is obtained by sampling DTFT of a signal at uniform frequency intervals.
- It converts continuous function of ω to a discrete function of ω
- So frequency analysis is possible by digital systems.
- $X(e^{j\omega})$ be discrete time Fourier transform of the discrete time signal x(n).
- The DFT of x(n) is obtained by sampling one period of the DTFT $X(e^{j\omega})$ at a finite number of frequency points.
- This is done at N equally spaced frequency points in the period $0\leq\omega\leq2\pi$
- The sampling frequency are denoted by ω_k

$$\omega_k = \frac{2\pi k}{N}$$

for k= 0,1,2,3,………., N-1

The sampling of $X(e^{j\omega})$ is mathematically expressed as

$X(k)= X(e^{j\omega})|_{\omega=\frac{2\pi k}{N}}$

for k= 0,1,2,3,………., N-1

Generally the DFT is defined along with number of samples and is called N-point DFT

Definition of DFT

Let x(n)=Discrete time signal of length L

X(k)=DFT of x(n)

The N-point DFT of x(n),where N L is defined as

$$X(k)= \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}}$$; for k= 0,1,2,3,………., N-1

Symbolically the N-point DFT of x(n) can be expressed as DFT{x(n)}=X(k)

Inverse DFT

The inverse DFT(IDFT) of the sequence X(k) of length N is defined as

$$x(n)= 1/N \sum_{k=0}^{N-1} X(k)e^{\frac{j2\pi kn}{N}}$$; for n= 0,1,2,3,………., N-1

<u>Properties of DFT</u>

- Linearity
- Periodicity
- Circular time shift
- Time reversal
- Multiplication
- Circular convolution

Linearity

Let DFT{$x_1(n)$}= $X_1(k)$ and DFT{$x_2(n)$}= $X_2(k)$ then by linearity property,

DFT{$a_1x_1(n)$ +$a_2x_2(n)$}= $a_1X_1(k)$+ $a_2X_2(k)$ where a_1 ,a_2 are constants

Periodicity

If a sequence x(n) is periodic with periodicity of N samples then N-point DFT,X(k) is also periodic with periodicity of N samples Hence ,if x(n) and X(k) are N point DFT pair then,

x(n+N)=x(n) ;for all n

X(k+N)=X(k) ;for all k

Circular time shift

This property says if a discrete time signal is circularly shifted in time by m units then its DFT is multiplied by $e^{\frac{-j2\pi km}{N}}$

i.e. if DFT{x(n)}=X(k),then DFT{$x((n-m))_N$}=X(k) $e^{\frac{-j2\pi km}{N}}$

Time reversal

This property says reversing the N-point sequence in time is equivalent to reversing the DFT sequence

i.e. if DFT{x(n)}=X(k), then DFT{x(N-n)}=X(N-k)

Multiplication

This property says that the DFT of product of two discrete time sequences is equivalent to circular convolution of the DFTs of the individual sequences scaled by a factor 1/N

i.e. if DFT{$x_1(n)$}= $X_1(k)$ and DFT{$x_2(n)$}= $X_2(k)$, then DFT{$x_1(n)$ $x_2(n)$}= 1/N [$X_1(k)$ * $X_2(k)$] * circular convolution

Circular convolution

This property says that the DFT of circular convolution of two sequences is equivalent to product of their individual DFTs

Let DFT{$x_1(n)$}= $X_1(k)$ and DFT{$x_2(n)$}= $X_2(k)$ then by convolution property

DFT{$x_1(n)$ * $x_2(n)$}= $X_1(k)$ $X_2(k)$

Relationship between DFT and Z-transform

The z transform of N-point sequence x(n) is given by, Z{x(n)}=X(z)= $\sum_{n=0}^{N-1} x(n)\, z^{-n}$

Let us evaluate X(z) at N equally spaced points on unit circle i.e. at z= $e^{\frac{-j2\pi k}{N}}$

$$x(z)= \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi kn}{N}}$$

We can conclude that the N-point DFT of a finite duration sequence can be obtained from the Z-transform of the sequence ,by evaluating the Z-transform of the sequence at N equally spaced points around the unit circle.

Compute the 4-point DFT of the sequence

x(n) =1/3 ; 0≤n≤2

=0 ;otherwise

x(n)={1/3,1/3,1/3}

4-point DFT (i.e. N=4)

$$X(k)= \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi kn}{N}}$$

$$X(k)= \sum_{n=0}^{3} x(n)\, e^{-\frac{j2\pi kn}{4}} = \sum_{n=0}^{2} x(n)\, e^{-\frac{j\pi kn}{2}}$$

$$= x(0)\, e^{0} + x(1)\, e^{\frac{-j\pi k}{2}} + x(2)\, e^{-j\pi k}$$

$=1/3 + 1/3\ e^{\frac{-j\pi k}{2}} \quad +1/3\ e^{-j\pi k}$

= 1/3[1+cos πk/2-jsin πk/2+cos πk-jsin πk]

So X(k)= 1/3[1+cos πk/2-jsin πk/2+cos πk-jsin πk]

- When k=0; X(0)= 1/3[1+cos 0-jsin 0 +cos 0-jsin 0]=1/3 [1+1+1]=1
- When k=1; X(1)= 1/3[1+cos π/2-jsin π/2+cos π-jsin π]=1/3 [1+0-j-1-j0]=-j/3
- When k=2; X(2)= 1/3[1+cos π-jsin π+cos 2π-jsin 2π]=1/3[1-1-j0+1-j0]=1/3
- When k=3; X(3)= 1/3[1+cos 3π/2-jsin 3π/2+cos 3π-jsin 3π]=1/3[1+0+j-1-j0]=j/3

X(k)={1,-j/3,1/3,j/3}

INTRODUCTION TO FAST FOURIER TRANSFORM(FFT) ALGORITHM

Direct Computation of DFT

- DFT of a sequence is obtained by direct computation. But this requires large number of computations which leads to greater processing time.
- DFT of a N-point sequence x(n) , n=0, 1, 2…………., N-1 is given by
  X(k)=$\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$ where k=0,1,2…………..,N-1.

- An N-point sequence yields an N-point transform.
- X(k) can be expressed as an inner product:
- X(k)= $[1 \quad e^{-j2\pi k/N} \quad e^{-j2\pi k2/N} e^{-j2\pi k3/N}$ .... $e^{-j2\pi k(N-1)/N}]$ $\begin{matrix} x(0) \\ x(1) \\ . \\ . \\ . \\ x(N-1) \end{matrix}$
- Notation: $W_N = e^{-j2\pi/N}$
- Hence,
- X(K)= $[1 \quad W_N\ K \quad \mathrm{WN}^{2K} \mathrm{WN}^{3K}$ .... $\mathrm{WN}^{(N-1)K}]$ $\begin{matrix} x(0) \\ x(1) \\ . \\ . \\ . \\ x(N-1) \end{matrix}$
- By varying k from 0 to N − 1 and combining the N inner products, X = Wx W is an N × N matrix, called as the "DFT Matrix".
- Each inner product requires N complex multiplications .There are N inner products Hence we require N^2 multiplications.

- Each inner product requires N − 1 complex additions. There are N inner products Hence we require N(N − 1) complex additions.
- If N is large then the number of computations will go into lakhs. This proves inefficiency of direct DFT computation.

Computationally efficient algorithm:FFT

- Fast Fourier Transform"(FFT) exploits the 2 important property (symmetry & periodicity) of W_N^K .
- FFT :Based on the fundamental principle of decomposing the computation of DFT of a sequence of length N into successively smaller DFTs.
- While calculating DFT, we have discussed N can be factorised as

  $N=r_1\ r_2\ r_3 \ldots\ldots\ldots r_v$ (Every r is a prime)

  If $r_1 = r_2 = r_3 = \ldots\ldots = r_v = r$ then $N=r^v$

  r is called the radix(base)of FFT algorithm and v indicates number of stages in FFT algorithm.
- **If r=2,it is called radix-2 FFT algorithm.**

  e.g if N=8 $=2^3$

For 8 point DFT there are 3 stages of FFT algorithm.

- Types of FFT algorithm

  1. Radix-2 Decimation In Time(DIT) algorithm
  2. Radix-2 Decimation In Frequency(DIF) algorithm

DIT ALGORITHM

Decimate means to "break into parts". DIT indicates dividing (splitting) the sequence in time domain.

First stage of Decimation

From {x(n)} form two sequences as follows: f1(n) =x(2n) and f2(n) =x(2n+1)

f1(n) contains the even-indexed samples, while f2(n) contains the odd-indexed samples

X(k)=$\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$ where k=0,1,2…………..,N-1

X(k)=$\sum_{n=0}^{N-1} x(n)\, W_N^{\ kn}$

X(k)=$\sum_{r=0}^{\frac{N}{2}-1} x(2r)\, W_N^{\ 2r*k} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1)\, W_N^{\ (2r+1)*k}$

X(K)=$\sum_{r=0}^{\frac{N}{2}-1} f\,1(n)\, W_N^{\ 2r*K} + W_N^{\ K}\sum_{r=0}^{\frac{N}{2}-1} f\,2(n)\, W_N^{\ (2r)*K}$

$W_N^{\ (2r)*k} = e^{-j2\pi k2r/N} = e^{-j2\pi kr/(N/2)} = W_{N/2}^{\ r*k}$

Hence

X(k)=$\sum_{r=0}^{\frac{N}{2}-1} f1\ W_{N/2}^{\ r*k} + W_N^{\ K}\sum_{r=0}^{\frac{N}{2}-1} f\,2(n)\, W_{N/2}^{\ r*k}$

=f1(k) + $W_N^{\ k}$ f2(k) where k=0,1,2…………..,N-1.

f1(k)and f2(k) are N/2 point DFTs

$W_N^{\ k}$is called "**Twiddle factor**" (Nth root of Unity)

- The N/2 point DFTs f1(k) and f2(k) are periodic with period N/2

  f1(k+N/2)=f1(k) (Periodicity Property)

f2(k+N/2)=f2(k) (Periodicity Property)

and$W_N^{k+N/2} = -W_N^k$ (Symmetry Property)

Hence, if X(k) = f1(k) + W_N^kf2(k), then X(k+N/2) = f1(k) - W_N^kf2(k),

f1(k) f1(k) + W_N^K f2(k)

f2(k) f1(k) - W_N^k f2(k)

Second Stage Decimation

Repeat the process for each of the sequences f1(n) and f2 (n).

f1(n) and f2 (n) will contain two N/4 point sequences each.

Let ,

v_{11} (n)= f1 (2n) & v_{12} (n)= f1 (2n+1)

where n=0,1,2,.......,N/4 -1

v_{21} (n)= f2(2n) & v_{22} (n)= f2 (2n+1)

Like earlier analysis we can show that,

$F1(k)=V_{11}(k) + W_{N/2}^k V_{12}(k)$

$F2(k)=V_{21}(k) + W_{N/2}^k V_{22}(k)$

Hence the N/2 point DFTs are obtained from the results of N/4 point DFTs

Summary of Steps of radix 2 DIT-FFT algorithm

- The no. of input samples, $N=2^M$,where M is an integer.
- The input sequence is shuffled through bit reversal.
- The no. of stages in the flow graph is given by $M=\log_2 N$
- Each stage consists of N/2 butterflies.

- The no of sets or sections of butterflies in each stage is given by 2^{M-m} .
- The twiddle factor exponents are given by $k=N*t/2^m$ where $t=0,1,2,\ldots\ldots 2^{m-1}-1$.
- Draw the flow graph taking N input samples and M stages.
- Calculate the DFT values using basic butterfly operations.

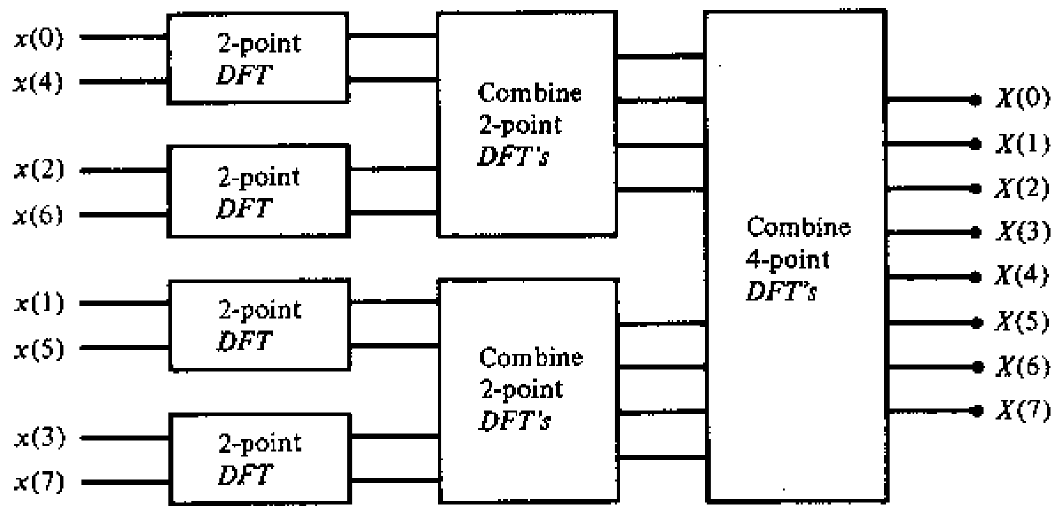
Recall that, for N = 8, the first split requires the data to be arranged as follows: x0, x2, x4, x6, x1, x3, x5, x7

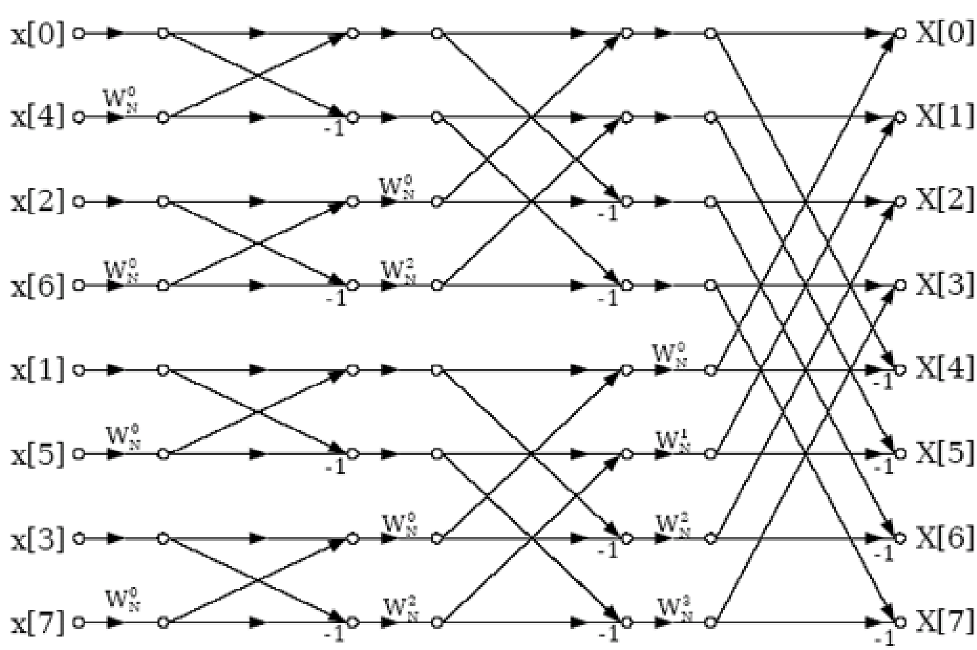
In the second and final split, the data appear in the following order: x0, x4, x2, x6, x1, x5, x3, x7

The final order is said to be in "bit reversed" form:

Original	Binary Form	Reversed Form	Final
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

3 stage computation of a 8 point DFT





Introduction to Digital Filters(FIR Filters)

- Digital Filters are the discrete time systems used mainly for filtering of arrays or sequences.
- The arrays or sequences. are obtained by sampling the input analog signals.

- Digital filters mainly performs frequency related operations such as Low Pass, High Pass, Band reject, Band pass and All pass.
- The design specifications include cut-off frequency, sampling frequency, stop band attenuation etc.
- Digital filters may be realised through hardware or software

Implementation of Digital Filters

- Represented by difference equations, implemented in s/w like ‘C’ or assembly language.
- Such languages are compiled and an executable code for the processor is prepared.
- This code runs on the memory, data bus ,shift registers, counters and ALU etc. to give required output.
- Digital Filters may also be implemented by dedicated hardware which is a digital circuit consisting of counters, shift registers, flip-flops, ALU etc.
- Disadvantage : we can perform only one type of filtering operation.

Types of Digital Filters

1.Finite Impulse Response (FIR) filters (non-recursive type)

2.Infinite Impulse Response (IIR) filters (recursive type)

- Basically ,digital filters are LTI systems which are characterised by unit sample response. The FIR system has finite duration unit sample response i.e.

  h(n)=0 for n<0 for n>=M
- Similarly IIR system has infinite duration unit sample response i.e.

  h(n)=0 for n<0

Introduction to Digital Signal Processor

- Microprocessors designed specifically for digital signal processing applications.
- Contains special architecture and instruction set to execute DSP algorithm efficiently.
- Types : 1.General Purpose DSPs 2. Special Purpose DSPs
- General Purpose DSPs: High speed microprocessor with architecture and instruction sets optimized for DSP operation. e.g. Texas Instruments TMS320C5x, TMS320C54x & Motorola DSP563x etc.
- Special Purpose DSPs: Contains h/w designed for specific DSP algorithms such as FFT, PCM, Filtering etc. e.g. FFT processor PDSP 16515A,TM 44/66, FIR filter UPDSP 16256 etc

Assignment Questions

1.Compute the 8 point DFT of the given sequence using radix-2 DIT FFT algorithm x(n)={1,3,1,2,1,3,1,2}.

2. Compute the 4 point DFT of the given sequence using radix-2 DIT FFT algorithm x(n)={1,1,2,2}.

2.What is phase factor or twiddle factor?

3.Draw and explain the basic butterfly diagram of DIT radix-2 FFT.

4. How many multiplications and additions are involved in radix-2 FFT?

Reference Books:

1.Signal and Systems by A Nagoor Kani

2.Digital Signal Processing by P. Ramesh Babu

3.Digital Signal Processing by Sanjay Sharma.

Reference site:

http://www.ee.iitm.ac.in/~csr/teaching/pg_dsp/lecnotes/fft.pdf

http://www.cmlab.csie.ntu.edu.tw/cml/dsp/training/coding/transform/fft.html